

# On torsors under elliptic curves and Serre's pro-algebraic structures

A. Bertapelle, J. Tong

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## Abstract

Let  $K$  be a local field with algebraically closed residue field and  $X_K$  a torsor under an elliptic curve  $J_K$  over  $K$ . Let  $X$  be a proper minimal regular model of  $X_K$  over the ring of integers of  $K$  and  $J$  the identity component of the Néron model of  $J_K$ . We study the canonical morphism  $q: \mathrm{Pic}_{X/S}^0 \rightarrow J$  which extends the biduality isomorphism on generic fibres. We show that  $q$  is pro-algebraic in nature with a construction that recalls Serre's work on local class field theory ([19]). Furthermore we interpret our results in relation to Shafarevich's duality theory for torsors under abelian varieties.

## Introduction

This paper concerns some local studies of torsors under an elliptic curve, or more generally, under an abelian variety. In the following, let  $\mathcal{O}_K$  be a complete discrete valuation ring with field of fractions  $K$  and algebraically closed residue field  $k$  of positive characteristic  $p > 0$ . Let  $\pi \in \mathcal{O}_K$  be a uniformizer of  $\mathcal{O}_K$ . Set  $S = \mathrm{Spec}(\mathcal{O}_K)$ , denote by  $s$  its closed point and by  $i: \mathrm{Spec}(k) \rightarrow S$  the usual closed immersion. Let  $J_K$  be an elliptic curve over  $K$  and  $J$  the identity component of its Néron model over  $S$ . Given a torsor  $X_K$  under  $J_K$  of order  $d$ , let  $X$  be its proper minimal regular model over  $S$ . In general  $X$  is not cohomologically flat in degree 0 over  $S$  (i.e., the canonical morphism  $k \rightarrow H^0(X_s, \mathcal{O}_{X_s})$  is not an isomorphism), and in particular the relative Picard functor  $\mathrm{Pic}_{X/S}^0$  is not representable, not even by an algebraic space. Nevertheless, the functor  $\mathrm{Pic}_{X/S}^0$  is not very far from being representable. For example, in [16] (see § 1.1.2 for a brief summary) it is shown that there exists an epimorphism of fppf-sheaves

$$q: \mathrm{Pic}_{X/S}^0 \rightarrow J$$

that extends the biduality isomorphism on generic fibres. This morphism plays a very important role in a recent work of Liu, Lorenzini and Raynaud ([11]) where, by considering the induced map  $\mathrm{Lie}(q)$  between the Lie algebras of  $\mathrm{Pic}_{X/S}^0$  and  $J$ , together with a result of T. Saito, the authors prove a beautiful result about the geometry of the scheme  $X$ , namely that the Kodaira type of the special fibre  $X_s$  of  $X$  is exactly  $d$  times the Kodaira type of the special fibre of the minimal regular  $S$ -model of the elliptic curve  $J_K$ .

One of the aims of this paper is to study the morphism  $q$  in order to reveal other interesting properties. More precisely, consider the surjective map induced by  $q$  on the  $S$ -sections (see § 1.1.2 for the surjectivity of  $q$ ):

$$(1) \quad q = q(S): \mathrm{Pic}^0(X) = \mathrm{Pic}_{X/S}^0(S) \rightarrow J(S).$$

Since the gcd of the multiplicities of the irreducible components of  $X_s$  is  $d$  (see Lemma 2.1.2), one finds that  $D := \frac{1}{d}X_s$  is a well-defined effective divisor of  $X$ , whose sheaf of ideals  $\mathcal{I} := \mathcal{O}_X(-D) \subset \mathcal{O}_X$  is invertible of order  $d$ , and generates the kernel of  $q$ . With the help of Greenberg realization functors, one can show that the morphism  $q$  is in fact *pro-algebraic* in nature, and we get a short exact sequence of *pro-algebraic groups* over  $k$  (see § 1.2):

$$(2) \quad 0 \rightarrow \mathbb{Z}/d\mathbb{Z} \rightarrow \mathbf{Pic}^0(\mathbf{X}) \xrightarrow{q} \mathbf{J}(\mathbf{S}) \rightarrow 0,$$

where the second map is given by sending  $\bar{1} \in \mathbb{Z}/d\mathbb{Z}$  to  $\mathcal{I} \in \mathbf{Pic}^0(\mathbf{X})(k) = \mathrm{Pic}^0(X)$ .

One of the main results of this paper shows that the morphism  $q$  in (1) can be thought as an analogue of the norm map studied by Serre in his work on local class field theory [19]. Let us first briefly review Serre's results. Let  $L/K$  be a finite Galois extension of  $K$  of Galois group  $\Gamma_{L/K}$ , and let  $U_K$  (respectively  $U_L$ ) be the group of units of the valuation ring  $\mathcal{O}_K$  of  $K$  (respectively  $\mathcal{O}_L$  of  $L$ ). Since the Brauer group  $\mathrm{Br}(K)$  is trivial ([5] 8.1 p. 203), the usual norm map

$$(3) \quad N_{L/K}: U_L \longrightarrow U_K$$

is surjective. By using the Greenberg realization functors, one can show that the morphism  $N_{L/K}$  is pro-algebraic in nature. This means first that each of the abstract groups  $U_K$  and  $U_L$  can be naturally endowed with a pro-algebraic structure. We will denote by  $\mathbf{U}_K$  and  $\mathbf{U}_L$  the pro-algebraic groups obtained in this way. Moreover, the norm morphism in (3) is the morphism on  $k$ -rational points induced by a morphism of pro-algebraic groups:

$$(4) \quad N_{L/K}: \mathbf{U}_L \longrightarrow \mathbf{U}_K.$$

On the other hand, the two pro-algebraic groups in (4) are naturally filtered: for each  $n \geq 1$ , one can define a pro-algebraic subgroup  $\mathbf{U}_K^n$  of  $\mathbf{U}_K$  whose group of  $k$ -rational points is given by the group  $U_K^n$  of  $n$ -units in  $K$ :

$$U_K^n(k) = U_K^n := \ker(U_K \rightarrow (\mathcal{O}_K/\pi^n \mathcal{O}_K)^\times)$$

Hence  $\mathbf{U}_K$  has the following filtration by its pro-algebraic subgroups:

$$\cdots \subset \mathbf{U}_K^{n+1} \subset \mathbf{U}_K^n \subset \cdots \subset \mathbf{U}_K^1 \subset \mathbf{U}_K^0 = \mathbf{U}_K.$$

Similarly,  $\mathbf{U}_L$  has the following filtration

$$\cdots \subset \mathbf{U}_L^{n+1} \subset \mathbf{U}_L^n \subset \cdots \subset \mathbf{U}_L^1 \subset \mathbf{U}_L^0 = \mathbf{U}_L.$$

In [19], Serre proved that these two filtrations are in fact compatible with respect to the norm map (4). More precisely, for each  $n \in \mathbb{Z}_{\geq 0}$ , the map  $N_{L/K}$  in (4) sends  $\mathbf{U}_L^{\psi(n)}$  onto  $\mathbf{U}_K^n$ , where  $\psi = \psi_{L/K}: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$  is the *Herbrand function* attached to the extension  $L/K$ , and used to define the upper numbering of the ramification filtration of the Galois group  $\Gamma_{L/K}$ .

In the situation of the present article, the two pro-algebraic groups in (2) are also naturally filtered (see § 3.1 for more details): the sheaf of ideals  $\mathcal{I} = \mathcal{O}_X(-D)$  allows us to define a  $\mathcal{I}$ -adic filtration on the first group  $\mathbf{Pic}^0(\mathbf{X})$ ; and on the other hand, the group  $\mathbf{J}(\mathbf{S})$  has a natural  $\pi$ -adic filtration. In § 3.3 we will explicitly compare these two filtrations. In order to get the right index in the comparison, we define and study in § 2 two numerical functions associated with the torsor  $X_K$  which can be thought as the analogue of the Herbrand functions used by

Serre in [19]. All these comparison results are explained in terms of the Greenberg realizations of  $\text{Pic}_{X/S}^0$  and  $J$  (see Theorem 3.4.3).

A second goal of this paper is to study the short exact sequence (2) in the framework of the duality theorems for abelian varieties. By using the short exact sequence (2), we get, for each torsor  $X_K$  of order  $d$ , an element of the group  $\text{Ext}_k^1(\mathbf{J}(\mathbf{S}), \mathbb{Z}/d\mathbb{Z})$  of extensions in the category of Serre pro-algebraic groups. More generally, considering torsors of order dividing  $d$  we can actually define a natural map of *sets* (see § 5 for details):

$$(5) \quad \Phi_d: {}_d\text{H}_{\mathbb{A}}^1(K, J_K) \rightarrow \text{Ext}_k^1(\mathbf{J}(\mathbf{S}), \mathbb{Z}/d\mathbb{Z}).$$

This construction is analogous to the one used in [19] to relate the Galois group  $\Gamma_K^{\text{ab}}$  of the maximal abelian extension of the field  $K$  with the fundamental group of the pro-algebraic group  $\mathbf{U}_K$ : namely, let  $L/K$  be a finite *abelian* extension of Galois group  $\Gamma_{L/K}$ , and let  $\mathbf{V}_L$  be the kernel of the norm map  $\mathbf{N}_{L/K}$  in (4). One then has the following short exact sequence of pro-algebraic groups

$$(6) \quad 0 \rightarrow \mathbf{V}_L \longrightarrow \mathbf{U}_L \xrightarrow{\mathbf{N}_{L/K}} \mathbf{U}_K \rightarrow 0.$$

The above sequence provides a homomorphism  $\pi_1(\mathbf{U}_K) \rightarrow \pi_0(\mathbf{V}_L)$  between the fundamental group of the pro-algebraic group  $\mathbf{U}_K$  and the group of connected components of  $\mathbf{V}_L$ . Moreover, there is a canonical isomorphism  $\tau: \pi_0(\mathbf{V}_L) \rightarrow \Gamma_{L/K}$  (cf. [19], 2.3). Now, the push-out of the sequence (6) via the composition of  $\tau$  with the canonical homomorphism  $\mathbf{V}_L \rightarrow \pi_0(\mathbf{V}_L)$  provides an element of  $\text{Ext}_k^1(\mathbf{U}_K, \Gamma_{L/K})$ , hence a homomorphism  $\pi_1(\mathbf{U}_K) \rightarrow \Gamma_{L/K}$ , where  $\Gamma_{L/K}$  coincides with its component group because it is a finite group. By passing to the limit on  $L$ , Serre obtained a homomorphism  $\theta: \pi_1(\mathbf{U}_K) \rightarrow \Gamma_K^{\text{ab}}$ . There exists then a homomorphism

$$\theta^*: \text{H}^1(K, \mathbb{Z}/d\mathbb{Z}) = \text{Hom}(\Gamma_K^{\text{ab}}, \mathbb{Z}/d\mathbb{Z}) \rightarrow \text{Hom}(\pi_1(\mathbf{U}_K), \mathbb{Z}/d\mathbb{Z}) = \text{Ext}_k^1(\mathbf{U}_K, \mathbb{Z}/d\mathbb{Z}),$$

which is in fact an isomorphism (cf. [19], 4.1). From this fact Serre deduced the main result of [19], namely that  $\theta$  is an isomorphism, and thus provided a “geometric” characterization of  $\Gamma_K^{\text{ab}}$ .

Now, our construction of the morphism  $\Phi_d$  appears to be an analogue of Serre’s construction. It then makes sense to ask if the map  $\Phi_d$  is an isomorphism too.

We then come to the second main result of this paper, which gives a new construction of Shafarevich’s pairing using the relative Picard functor. Since this discussion also holds for abelian varieties of higher dimension, let us, more generally, consider an abelian  $K$ -variety,  $A_K$  and a torsor  $X_K$  under  $A_K$  of order  $d$ . It is known that it is still possible to associate to  $X_K$  an extension of the pro-algebraic group  $\mathbf{Gr}(A^0)$  by  $\mathbb{Z}/d\mathbb{Z}$  and that this construction, provides an isomorphism

$$(7) \quad \text{H}_{\mathbb{A}}^1(K, A_K) \xrightarrow{\sim} \text{Ext}(\mathbf{Gr}(A^0), \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\pi_1(\mathbf{Gr}(A^0)), \mathbb{Q}/\mathbb{Z}), \quad (\text{Shafarevich duality})$$

where  $A^0$  is the identity component of the Néron model of the dual abelian variety and  $\mathbf{Gr}(A^0)$  denotes its perfect Greenberg realization (see § 1.2). The result in (7) was proved by Shafarevich for the prime-to- $p$  parts, by Bégueri in [1] for  $K$  of characteristic 0 and by Bester and the first author, respectively in [4] and [2], for the equal positive characteristic case; there are also results of Vvedenskiĭ on elliptic curves. In the fourth section of the paper, after recalling the construction of Shafarevich’s duality (7) in the case of mixed characteristic, we slightly modify Bégueri’s construction using rigidifiers. The latter construction works in any characteristic. Then we construct a morphism as in (7) via the relative Picard functor and we show that it always coincides with the modified Bégueri construction and thus with Shafarevich’s duality for

$K$  of characteristic 0. In the characteristic  $p$  case it coincides with Shafarevich's duality on the prime-to- $p$  parts. The analogous result for the  $p$  parts, although expected, is still open.

In the fifth section of the paper, as a direct corollary of the previous study of the Shafarevich's pairing, we show that the map  $\Phi_d$  in (5), only defined for  $J_K$  an elliptic curve, is a morphism of groups. Furthermore if  $d$  is prime to  $p$ , or with no restriction on  $d$  in the mixed characteristic case,  $\Phi_d$  is the restriction of (7) to the  $d$ -torsion subgroups. In particular it is an isomorphism. This is done by showing that, in the case of elliptic curves, the short exact sequence (2) coincides with the sequence associated with  $X_K$  via the new construction of (7) given in § 4.

This paper arises from the confluence and the comparison of the results contained in the preprints [3] and [22].

## 1 The Picard functor and the Greenberg functor

In this section we recall well-known results on Picard functors and Greenberg functors. As usual, we denote by  $\mathbf{Sch}/X$  the category of  $X$ -schemes and by  $\mathfrak{Ab}$  the category of abelian groups. For simplicity, we will assume in the following that  $S = \mathrm{Spec}(\mathcal{O}_K)$  is the spectrum of a discrete valuation ring  $\mathcal{O}_K$ , with  $K$  the fraction field of  $\mathcal{O}_K$  and  $k$  its residue field.

### 1.1 The Picard functor

#### 1.1.1 Definitions and first results

Let  $f: X \rightarrow S$  be a proper morphism of schemes. We denote by

$$\mathrm{Pic}_{X/S}: \mathbf{Sch}/S \rightarrow \mathfrak{Ab}$$

the *relative Picard functor of  $X$  over  $S$* , i.e., the fppf sheaf associated with the presheaf  $S' \mapsto \mathrm{Pic}(X \times_S S')$ . It is also the sheaf for the étale topology associated with the presheaf  $S' \mapsto \mathrm{Pic}(X \times_S S')$  ([16], 1.2).

Suppose in the following that  $f$  is proper and flat. In general the functor  $\mathrm{Pic}_{X/S}$  is not representable. It is representable by an algebraic  $S$ -space if and only if  $X/S$  is *cohomologically flat (in degree 0)*, i.e., if the formation of the direct image  $f_*\mathcal{O}_X$  commutes with base change. Even when  $\mathrm{Pic}_{X/S}$  is not representable, it has a nice presentation by algebraic  $S$ -spaces. To see this fact, we recall the notion of rigidificator.

**Definition 1.1.1** ([5] 8.1/5). Let  $i: Y \hookrightarrow X$  be a closed immersion of  $S$ -schemes with  $Y$  finite and flat over  $S$ . One says that  $(Y, i)$  is a *rigidificator* of  $\mathrm{Pic}_{X/S}$  if the following condition holds: for any  $S$ -scheme  $S'$ , if  $i': Y' \rightarrow X'$  denotes the morphism obtained from  $i$  after base change along  $S' \rightarrow S$ , the map

$$\Gamma(i'): \Gamma(X', \mathcal{O}_{X'}) \rightarrow \Gamma(Y', \mathcal{O}_{Y'})$$

is injective.

In the sequel let  $f: X \rightarrow S$  be proper and flat, and let  $(Y, i)$  be a rigidificator of  $\mathrm{Pic}_{X/S}$ ; it exists by the hypothesis on  $f$  ([16] proposition 2.2.3 (c)). For any scheme  $S'$  over  $S$ , an *invertible sheaf on  $X' = X \times_S S'$ , rigidified along  $Y'$* , is a pair  $(\mathcal{L}, \alpha)$ , where  $\mathcal{L}$  is an invertible sheaf on  $X'$  and  $\alpha: \mathcal{O}_{Y'} \simeq i'^*\mathcal{L}$  is an isomorphism (i.e.,  $\alpha$  is a trivialization of  $i'^*\mathcal{L}$ ). An isomorphism between two rigidified invertible sheaves  $(\mathcal{L}, \alpha), (\mathcal{M}, \beta)$ , on  $X'$  is an isomorphism of  $\mathcal{O}_{X'}$ -modules

$u: \mathcal{L} \rightarrow \mathcal{M}$  such that the following diagram commutes:

$$\begin{array}{ccc} i'^*\mathcal{L} & \xrightarrow{i'^*u} & i'^*\mathcal{M} \\ & \searrow \alpha \quad \nearrow \beta & \\ & \mathcal{O}_{Y'} & \end{array}$$

Denote then by  $(\text{Pic}_{X/S}, Y)(S')$  the set of isomorphism classes of invertible sheaves on  $X'$  rigidified along  $Y'$ . As  $S'$  varies in the category of  $S$ -schemes  $(\mathbf{Sch}/S)$ , the association  $S' \mapsto (\text{Pic}_{X/S}, Y)(S')$  defines a functor of abelian groups  $(\text{Pic}_{X/S}, Y)$ , called *the rigidified Picard functor of  $X/S$  relative to the rigidificator  $Y$* . Concerning its representability we have:

**Theorem 1.1.2** ([16], 2.3.1 & 2.3.2). *The functor  $(\text{Pic}_{X/S}, Y)$  is representable by an algebraic space over  $S$ , locally of finite presentation. Furthermore, if  $X/S$  is a curve, the algebraic space  $(\text{Pic}_{X/S}, Y)$  is smooth over  $S$ .*

One has a canonical morphism of sheaves of groups  $r: (\text{Pic}_{X/S}, Y) \rightarrow \text{Pic}_{X/S}$ , that forgets the rigidification. Étale locally, any element in  $\text{Pic}_{X/S}(S')$  is represented by an invertible sheaf on  $X'$  such that its pull-back to  $Y' := Y \times_S S'$  is trivial (this is possible since  $Y$  is finite). Hence  $r$  is an epimorphism for the étale topology. To study the kernel of  $r$ , denote by  $V_X^*$  (respectively by  $V_Y^*$ ) the fppf sheaf on  $\mathbf{Sch}/S$ , given by  $S' \mapsto \Gamma(X', \mathcal{O}_{X'})^*$  (respectively by  $S' \mapsto \Gamma(Y', \mathcal{O}_{Y'})^*$ ). They are representable by  $S$ -schemes (cf. [16], 2.4.0). By definition of rigidificator the natural map  $V_X^* \rightarrow V_Y^*$  is injective. Let  $u: V_Y^* \rightarrow (\text{Pic}_{X/S}, Y)$  be the map defined as follows on  $S'$ -sections,  $S'$  an  $S$ -scheme:

$$a \in V_Y^*(S') = \Gamma(Y_{S'}, \mathcal{O}_{Y_{S'}}^*) \mapsto (\mathcal{O}_{X_{S'}}, \alpha_a) \in (\text{Pic}_{X/S}, Y)(S')$$

where  $\alpha_a: \mathcal{O}_{Y_{S'}} \rightarrow \mathcal{O}_{Y_{S'}} = \mathcal{O}_{X_{S'}|Y_{S'}}$  is the multiplication by  $a$ . Clearly  $\text{im}(u) \subset \ker(r)$  and thus one obtains a complex of fppf sheaves over  $S$ :

$$(8) \quad 0 \longrightarrow V_X^* \longrightarrow V_Y^* \xrightarrow{u} (\text{Pic}_{X/S}, Y) \xrightarrow{r} \text{Pic}_{X/S} \longrightarrow 0,$$

which is exact for the étale topology ([16], 2.1.2(b), 2.4.1).

### 1.1.2 Néron models and Picard functors

We suppose in this section that the discrete valuation ring  $\mathcal{O}_K$  is strictly henselian *with algebraically closed residue field*. In particular, the Brauer group  $\text{Br}(K) = 0$ . Let  $f: X \rightarrow S$  be a proper and flat curve with geometrically connected fibres. Denote by  $P$  (respectively by  $(P, Y)$ ) the open subfunctor of  $\text{Pic}_{X/S}$  (respectively of  $(\text{Pic}_{X/S}, Y)$ ) consisting of invertible sheaves of total degree 0 (respectively of invertible sheaves of total degree 0 rigidified along  $Y$ ). Then  $(P, Y)$  is an open algebraic subspace of  $(\text{Pic}_{X/S}, Y)$ , and  $P$  is the schematic closure of  $(\text{Pic}_{X/S})_K^0$  in  $\text{Pic}_{X/S}$ , while  $(P, Y)$  is the schematic closure of  $(\text{Pic}_{X/S}, Y)_K^0$  in  $(\text{Pic}_{X/S}, Y)$ . Denote by  $E$  the schematic closure of the unit section of  $P_K$  in  $P$ , and define  $Q$  as the fppf quotient of  $P$  by  $E$ . It is the biggest separated quotient of  $P$ . It is representable by a separated group scheme over  $S$  ([16], 3.3.1). Denote by  $q$  the canonical map

$$(9) \quad q: P \rightarrow Q;$$

it is surjective for the fppf topology, and it induces an isomorphism on generic fibres since  $E_K$  is the unit section of  $P_K = \text{Pic}_{X_K/K}^0$ .

**Theorem 1.1.3** ([11], 3.7). *Let  $f: X \rightarrow S$  be a proper and flat curve with  $X$  regular and  $f_*\mathcal{O}_X = \mathcal{O}_S$ . Then the group scheme  $\mathcal{Q}/S$  is the Néron model of  $P_K = \text{Pic}_{X_K/K}^0$ .*

Suppose then  $X$  regular. Denote by  $J = \mathcal{Q}^0$  the identity component of  $\mathcal{Q}$ , and by  $\text{Pic}_{X/S}^0$  the subfunctor of  $P \subset \text{Pic}_{X/S}$  given by the invertible sheaves of degree 0 on each irreducible component of  $X$ . Since the functor  $\text{Pic}_{X/S}^0$  has connected fibres, the restriction to  $\text{Pic}_{X/S}^0$  of  $q$  in (9) factors through the identity component  $J$  of  $\mathcal{Q}$ . By abuse of notation, we will use the same letter  $q$  to denote the induced map:

$$(10) \quad q: \text{Pic}_{X/S}^0 \rightarrow J.$$

Finally, since  $\text{Br}(K) = 0$ , the following map

$$(P, Y)^0(S) = (\text{Pic}_{X/S}, Y)^0(S) \longrightarrow J(S),$$

induced by the morphism  $r$  in (8) is surjective (see the proof of 9.6/1 of [5]); hence so too is the morphism (again denoted by  $q$ ) induced by (10):

$$(11) \quad q = q(S): P^0(S) = \text{Pic}_{X/S}^0(S) \longrightarrow J(S).$$

## 1.2 Pro-algebraic groups and Greenberg functor

From now on, we suppose that the discrete valuation ring  $\mathcal{O}_K$  is complete with algebraically closed residue field  $k$ . In the following, a *pro-algebraic group over  $k$*  is a pro-object in the category of  $k$ -group schemes of finite type (see [14], [12]). This notion, does not coincide with the notion of pro-algebraic groups introduced by Serre in [18], where the author considers the category of  $k$ -group schemes, but up to purely inseparable morphisms. Since we use both categories, we call the objects of the latter category Serre pro-algebraic groups, and denote them with bold letters.

Let  $G$  be a smooth group scheme of finite type over  $S$ . The Greenberg functors allow us to construct a pro-algebraic group over  $k$  associated with  $G$  whose group of  $k$ -points is  $G(R)$ . Let us recall the construction. We denote by  $W$  the ring of Witt vectors of  $k$  and by  $\mathbb{W}$  the Witt functor on the category of  $k$ -algebras  $\mathbf{Alg}/k$ . Let  $n \in \mathbf{Z}_{\geq 1}$  and  $\mathcal{O}_{K,n} = \mathcal{O}_K/\pi^n$ . Then  $\mathcal{O}_{K,n}$  is canonically a  $W$ -module of finite length. Let  $\mathbb{R}_n$  be its associated Greenberg algebra (see [10], Appendix A), which is by definition the fpqc sheaf on  $\mathbf{Alg}/k$  associated with the pre-sheaf:  $A \mapsto \mathcal{O}_{K,n} \otimes_W \mathbb{W}(A)$ . One defines then  $\text{Gr}_n(G)$  as the sheaf on  $\mathbf{Alg}/k$  given by  $A \mapsto G(\mathbb{R}_n(A))$ . It is representable by a smooth  $k$ -group scheme of finite type ([7]). For any  $n \geq 1$ , the canonical map  $\mathcal{O}_{K,n+1} \rightarrow \mathcal{O}_{K,n}$  induces a smooth morphism of  $k$ -group schemes  $\alpha_n: \text{Gr}_{n+1}(G) \rightarrow \text{Gr}_n(G)$ , whose kernel is a connected unipotent  $k$ -group scheme. Furthermore the canonical map  $G(\mathcal{O}_{K,n}) \rightarrow \text{Gr}_n(G)(k)$  is an isomorphism. Thanks to this identification the morphism  $\alpha_n(k): \text{Gr}_{n+1}(G)(k) \rightarrow \text{Gr}_n(G)(k)$  is identified with the canonical morphism  $G(\mathcal{O}_{K,n+1}) \rightarrow G(\mathcal{O}_{K,n})$ . The algebraic  $k$ -groups  $\text{Gr}_n(G)$  form then a projective system  $\{(\text{Gr}_n(G), \alpha_n)\}_{n \geq 1}$  of algebraic  $k$ -groups with smooth transition maps having connected kernels. We will denote it by  $\text{Gr}(G)$  and call it the Greenberg realization of  $G$ . Furthermore, as explained in [12], III § 4, if we consider the perfect group schemes  $\mathbf{Gr}_n(G)$  associated with the  $k$ -group schemes  $\text{Gr}_n(G)$ , they are quasi-algebraic groups in the sense of [18], 1.2, and hence the projective system  $\mathbf{Gr}(G) = \{(\mathbf{Gr}_n(G), \alpha_n)\}_{n \geq 1}$  determines a pro-algebraic group in the sense of Serre; we will call it the perfect Greenberg realization of  $G$ . The group of  $k$ -rational points of  $\mathbf{Gr}(G)$  is  $G(S)$ . For this reason sometimes we denote it by  $\mathbf{G}(S)$ . Observe that this construction also works for any smooth  $S$ -scheme  $G$ , but  $\mathbf{Gr}(G)$  may not be a Serre pro-algebraic group, since its component group may not be profinite: for



example, consider as  $G$  the Néron model of  $\mathbb{G}_{m,K}$ ; then  $\mathbf{Gr}(G)$  is the projective limit of perfect  $k$ -schemes having component group  $\mathbb{Z}$ .

In the category of Serre pro-algebraic groups, the component group functor  $\pi_0$  admits a left derived functor  $\pi_1$  which is left exact. We list below some well-known facts used in this paper. By simply assuming them, the reader unfamiliar with the theory of pro-algebraic groups will be able to follow the proofs.

- i) If  $Y$  is a smooth  $S$ -group scheme of finite type, then  $\mathbf{Gr}(Y^0) = \mathbf{Gr}(Y)^0$ ,  $\pi_0(\mathbf{Gr}(Y))$  coincides with the component group of the special fibre  $Y_s$  and  $\pi_1(\mathbf{Gr}(Y))$  is a profinite group.
- ii) If  $\mathbf{P}$  is a Serre pro-algebraic group and  $\mathbf{P}^0$  is its identity component, then  $\pi_1(\mathbf{P}) = \pi_1(\mathbf{P}^0)$ .
- iii) A short exact sequence of smooth  $S$ -group schemes of finite type  $0 \rightarrow Y_1 \rightarrow Y_2 \rightarrow Y_3 \rightarrow 0$  provides a long exact sequence of profinite groups (cf. [18], 10.2/1)

$$\begin{aligned} 0 \rightarrow \pi_1(\mathbf{Gr}(Y_1)) \rightarrow \pi_1(\mathbf{Gr}(Y_2)) \rightarrow \pi_1(\mathbf{Gr}(Y_3)) \rightarrow \\ \rightarrow \pi_0(\mathbf{Gr}(Y_1)) \rightarrow \pi_0(\mathbf{Gr}(Y_2)) \rightarrow \pi_0(\mathbf{Gr}(Y_3)) \rightarrow 0. \end{aligned}$$

### 1.2.1 Greenberg functor and Picard functor

Let  $f: Z \rightarrow S$  be a proper morphism of schemes with  $Z$  of dimension one and  $f(Z) = \{s\}$ , *e.g.* for  $X/S$  a flat curve, one can take  $Z = X \times_S \mathrm{Spec}(\mathcal{O}_{K,n})$  viewed as an  $S$ -scheme. The map  $f$  is never flat and the Picard functors  $\mathrm{Pic}_{Z/S}$  and  $\mathrm{Pic}_{Z/S}^0$  are not representable. However, as shown by Lipman in [10], the Greenberg realization of the sheaf  $\mathrm{Pic}_{Z/S}$  (respectively of  $\mathrm{Pic}_{Z/S}^0$ ) is represented by a smooth  $k$ -group scheme. More precisely, one takes  $\mathrm{Gr}(\mathrm{Pic}_{Z/S})$  (respectively  $\mathrm{Gr}(\mathrm{Pic}_{Z/S}^0)$ ) as the fppf sheaf associated with the presheaf ([10], 1.8)

$$\mathbf{Alg}/k \rightarrow \mathfrak{Ab}, \quad \mathbf{A} \mapsto \mathrm{Pic}_{Z/S}(\mathbb{R}_n(\mathbf{A})) \quad (\text{resp. } \mathbf{A} \mapsto \mathrm{Pic}_{Z/S}^0(\mathbb{R}_n(\mathbf{A}))).$$

where, as above,  $\mathbb{R}_n$  denotes the Greenberg algebra associated with  $\mathcal{O}_{K,n}$ .

**Theorem 1.2.1** ([10]). *Let  $f: Z \rightarrow S$  be as above. Then the functor  $\mathrm{Gr}(\mathrm{Pic}_{Z/S})$  (respectively  $\mathrm{Gr}(\mathrm{Pic}_{Z/S}^0)$ ) is representable by a smooth  $k$ -group scheme (respectively by a smooth and connected  $k$ -group scheme), whose dimension is equal to the length of the  $W$ -module  $H^1(Z, \mathcal{O}_Z)$ . Furthermore, the canonical morphisms*

$$\mathrm{Pic}(Z) \rightarrow \mathrm{Gr}(\mathrm{Pic}_{Z/S})(k), \quad \mathrm{Pic}^0(Z) \rightarrow \mathrm{Gr}(\mathrm{Pic}_{Z/S}^0)(k)$$

*are isomorphisms.*

## 2 Herbrand functions

The content of the next two sections is mainly from the preprint [22]. From now on, let  $K$  be a local field with algebraically closed residue field  $k$  and denote by  $\mathcal{O}_K$  its ring of integers. Let  $A_K$  be an elliptic curve,  $X_K$  a torsor under  $A_K$  over  $K$ , and  $f: X \rightarrow S$  a flat proper curve such that the following conditions are satisfied: (i)  $f_*(\mathcal{O}_X) = \mathcal{O}_S$ ; (ii) The generic fibre of  $X/S$  is given by the torsor  $X_K/K$ , hence is a geometrically connected algebraic curve, which is smooth of arithmetic genus 1 (*i.e.*,  $\dim_K H^1(X_K, \mathcal{O}_{X_K}) = 1$ ); (iii)  $X$  is a minimal regular surface over  $S$ . Let  $X_s = \sum_{i=1}^r n_i C_i$  be the decomposition of the special fibre  $X_s$  into the sum of its reduced

irreducible components, and denote by  $d$  the gcd of the integers  $n_i$ . Moreover, let  $D$  be the divisor of  $X$  given by

$$(12) \quad \frac{1}{d}X_s = \sum_{i=1}^r \frac{n_i}{d}C_i,$$

and let  $\mathcal{I}$  be the ideal sheaf of  $D$ . The special fibre  $X_s$  of  $X$  is then defined by the ideal sheaf  $\mathcal{I}^d = \pi\mathcal{O}_X \subset \mathcal{O}_X$ . For each  $n \in \mathbf{Z}_{\geq 1}$ , let  $X_n$  be the closed subscheme of  $X$  defined by the ideal  $\mathcal{I}^n \subset \mathcal{O}_X$ . Hence,  $X_n$  is the closed subscheme of  $X$  which defines the divisor  $nD$  of  $X$ . The aim of this section is to study the variation of the numerical function  $n \mapsto h^1(X_n, \mathcal{O}_{X_n})$  ( $:=$  the length of the  $\mathcal{O}_K$ -module  $H^1(X_n, \mathcal{O}_{X_n})$ ).

We recall the following useful result.

**Lemma 2.0.2** ([13], p. 332). *Let  $L$  be an invertible sheaf over  $X_1$ , which is of degree 0 on each component of  $X_1$ . Then, if  $H^0(X_1, L) \neq 0$ , we have  $L \simeq \mathcal{O}_{X_1}$  and  $H^0(X_1, \mathcal{O}_{X_1}) \simeq k$ .*

## 2.1 Study of the dualizing sheaf

We begin with two classical results.

**Lemma 2.1.1.** *With the notation as above, and in particular for  $J_K = \text{Pic}_{X_K/K}^0$ .*

- (1) *There is a canonical isomorphism of elliptic curves  $\iota: A_K^\vee \simeq J_K$ .*
- (2) *Let  $\sigma: A_K \rightarrow J_K$  be the isomorphism obtained by composing the isomorphism  $\alpha: A_K \simeq A_K^\vee$  sending  $a \in A_K$  to  $\mathcal{O}_{A_K}(a - o)$  (here  $o \in A_K$  is the neutral element), with the isomorphism  $\iota$  in (1). Then the canonical map  $\psi: X_K \rightarrow \text{Pic}_{X_K/K}^1$  sending  $x$  to  $\mathcal{O}_{X_K}(x)$  is equivariant with respect to the isomorphism  $\sigma$ , where  $\text{Pic}_{X_K/K}^1$  is the Picard scheme which classifies the line bundles of degree 1 on  $X_K$ . In particular, under the identification given by  $\sigma$ , the torsor  $X_K \in H_{\text{fl}}^1(K, A_K)$  is equal to the torsor  $\text{Pic}_{X_K/K}^1$  in  $H_{\text{fl}}^1(K, J_K)$ .*

*Proof.* The first fact is proved in [17], XIII, 1.1. For the second fact, in order to verify that the morphism  $\psi$  is equivariant with respect to the morphism  $\sigma$ , by descent, we need only prove the corresponding statement over a separable closure  $\bar{K}$  of  $K$ . Over  $\bar{K}$ , the isomorphism  $\sigma: A_{\bar{K}} \rightarrow J_{\bar{K}}$  can be explicitly described by mapping  $a \in A_{\bar{K}}$  to  $\mathcal{O}_{X_{\bar{K}}}(a \cdot x_0 - x_0)$  with  $x_0 \in X_K(\bar{K})$ . Hence to show the desired property of (2), we are reduced to proving the following equality between line bundles on  $X_{\bar{K}}$ : for any  $x \in X_{\bar{K}}(\bar{K})$ , and any  $a \in A_{\bar{K}}(\bar{K})$  we have  $\mathcal{O}_{X_{\bar{K}}}(a \cdot x) \simeq \mathcal{O}_{X_{\bar{K}}}(a \cdot x_0 - x_0) \otimes \mathcal{O}_{X_{\bar{K}}}(x)$ , or equivalently,  $\mathcal{O}_{X_{\bar{K}}}(a \cdot x_0 - x_0) \simeq \mathcal{O}_{X_{\bar{K}}}(a \cdot x - x)$ . Indeed, since the isomorphism  $\iota_{\bar{K}}$  is independent of the choice of  $x_0$ , we have  $\sigma(a) = \iota_{\bar{K}, x_0}(\alpha(a)) = \mathcal{O}_{X_{\bar{K}}}(a \cdot x_0 - x_0) = \iota_{\bar{K}, x}(\alpha(a)) = \mathcal{O}_{X_{\bar{K}}}(a \cdot x - x)$ .  $\square$

**Lemma 2.1.2.** *Let  $d_1$  be the order of the torsor  $X_K$  in the group  $H_{\text{fl}}^1(K, J_K)$ , and  $d_2$  be the minimal degree of extensions  $K'$  of  $K$  such that  $X_K(K') \neq \emptyset$ . We also define  $d_3$  as the minimum of the multiplicities of the irreducible components of  $X_s$ . Then we have  $d_1 = d_2 = d_3 = d$ , with  $d$  as in (12).*

*Proof.* We will prove this Lemma by showing that  $d \leq d_1 \leq d_2 \leq d_3 \leq d$ . Let  $n \in \mathbf{Z}_{>0}$  be a positive integer, by Lemma 2.1.1 (2), the torsor  $nX_K$  is isomorphic to the irreducible component  $\text{Pic}_{X_K/K}^n$  of  $\text{Pic}_{X_K/K}$  which classifies the invertible sheaves of degree  $n$ . Hence, the torsor  $nX_K$  is trivial if and only if  $\text{Pic}_{X_K/K}^n(K) \neq \emptyset$ . On the other hand, since  $\mathcal{O}_K$  is strictly henselian with algebraically closed residue field, we have  $\text{Br}(K) = 0$ . Hence,  $\text{Pic}_{X_K/K}^n(K) = \text{Pic}^n(X_K)$ . As a



result,  $d_1$  is also the minimum of the degrees of the divisors with positive degree on  $X_K$ . Now, let  $\Sigma_K \subset X_K$  be any divisor with positive degree, and let  $\Sigma$  be its schematic closure in  $X$ . Then we have  $\deg(\Sigma_K) = \Sigma \cdot X_s = \Sigma \cdot (dD) = d(\Sigma \cdot D)$ . Hence  $d \mid \deg(\Sigma_K)$ , in particular  $d \leq \deg(\Sigma_K)$ . As a result, we get  $d \leq d_1$ . Next, by definition of  $d_2$ , there exists a closed point of degree  $d_2$  on  $X_K$ , hence a divisor of degree  $d_2$  on  $X_K$ , so we have  $d_1 \leq d_2$ . Since  $\mathcal{O}_K$  is strictly henselian, for each  $i$ , we can find a *positive* divisor  $\Delta_i$  of  $X/S$  of degree  $n_i$  ([5] 9.1/10). In particular, we have  $d_2 \leq n_i$  for each  $i$ , hence we get  $d_2 \leq d_3$ . Finally, to see that  $d_3 \leq d$ , note that a suitable combination of  $\Delta_i$  gives us a divisor  $\Delta'$  of degree  $d$  of  $X_K$ . In general, the divisor  $\Delta'$  might be not positive, but since  $X_K$  is of arithmetic genus 1, and  $d \geq 1$ , we have  $h^0(X_K, \mathcal{O}_{X_K}(\Delta'_K)) > 0$ . Hence there exists hence a *positive* divisor  $\Delta_K$  of degree  $d$  of  $X_K$  which is linearly equivalent to  $\Delta'_K$ . Let  $\Delta = \overline{\Delta_K}$  the schematic closure of  $\Delta_K$  in  $X$ , we have

$$d = \deg(\Delta_K) = \Delta \cdot X_s = d(\Delta \cdot D).$$

In particular,  $\Delta \cdot D = 1$ , and  $\Delta \cap D = \{y\}$  consists of only one point, and  $D$  is *regular* at  $y$ . Let  $C_i$  be the irreducible component of  $D$  passing through  $y$ , then  $C_i$  is of multiplicity  $d$  in  $X_s$ , whence  $d_3 \leq d$ . This completes the proof.  $\square$

We denote by  $\omega_{X/S} = f^! \mathcal{O}_S$  the *relative dualizing sheaf* of  $f: X \rightarrow S$ , and for any  $n \geq 0$ , let  $\omega_n$  be the dualizing sheaf of  $X_n/S$ . Hence  $\omega_n = (\mathcal{O}_X(X_n) \otimes \omega_{X/S})|_{X_n}$ ; we denote with the same symbol the corresponding divisor on  $X$ . Since  $X$  is regular,  $X/S$  is local complete intersection and the dualizing sheaf  $\omega_{X/S}$  is *invertible*. Moreover, since  $X_K/K$  is smooth projective of genus 1 and the formation of the dualizing sheaf is compatible with base change, one finds that  $(\omega_{X/S})_K \simeq \mathcal{O}_{X_K}$ . On the other hand, according to the Grothendieck-Serre duality theorem, for any coherent sheaf  $\mathcal{F}$  on  $X_n$ , we have the following canonical isomorphisms:

$$H^0(X_n, \mathcal{F}^\vee \otimes \omega_n) \simeq \text{Ext}_{\mathcal{O}_K}^1(H^1(X_n, \mathcal{F}), \mathcal{O}_K).$$

Since the  $\mathcal{O}_K$ -module  $H^1(X_n, \mathcal{F})$  is killed by  $\pi^n$ , we also have

$$\text{Ext}_{\mathcal{O}_K}^1(H^1(X_n, \mathcal{F}), \mathcal{O}_K) \simeq \text{Hom}_{\mathcal{O}_K}(H^1(X_n, \mathcal{F}), \mathcal{O}_K/\pi^m \mathcal{O}_K).$$

for any integer  $m \geq n$ .

**Lemma 2.1.3.** *For any  $i = 1, \dots, r$ , we have  $\omega_{X/S} \cdot C_i = 0$ .*

*Proof.* Since  $\omega_{X/S}|_{X_K} \simeq \mathcal{O}_{X_K}$ , we have  $\omega_{X/S} \cdot X_s = 0$ , i.e.,  $\sum_{i=1}^r n_i (\omega_{X/S} \cdot C_i) = 0$ . In particular, we get the Lemma if  $r = 1$ . Suppose now  $r \geq 2$ ; since  $C_i \cdot X_s = 0$ , we obtain  $C_i \cdot C_i < 0$ . If  $\omega_{X/S} \cdot C_i < 0$ , since  $2g(C_i) - 2 = (\omega_{X/S} + C_i) \cdot C_i \geq -2$ , we have  $g(C_i) = 0$ ,  $C_i \cdot C_i = -1$ . This gives us a contradiction with the fact that  $X/S$  is a minimal regular surface. So  $(\omega_{X/S} \cdot C_i) \geq 0$ . As a result,  $\omega_{X/S} \cdot C_i = 0$  for any  $i$ .  $\square$

**Corollary 2.1.4.** *There is a unique integer  $n$ ,  $0 \leq n < d$ , such that  $\omega_{X/S} \simeq \mathcal{I}^n$ .*

*Proof.* Since  $\omega_{X/S}|_{X_K} \simeq \mathcal{O}_{X_K}$ ,  $\omega_{X/S} \simeq \mathcal{O}_X(Y)$ , with  $Y$  a divisor of  $X$  with support contained in  $X_s$ . Hence,  $Y$  is a combination of the components  $C_i$ . On the other hand, according to the previous Lemma,  $Y \cdot C_i = 0$ , we have  $Y \cdot Y = 0$ , hence,  $Y$  is a rational multiple of  $X_s$  ([5], 9.5/10), i.e.,  $Y$  is linearly equivalent to  $nD$  with  $0 \leq n < d$ , whence the Corollary follows.  $\square$

**Corollary 2.1.5.** *Suppose that  $f: X \rightarrow S$  has a section  $\sigma$  defined by the ideal sheaf  $\mathcal{J}$ . Set  $\omega = \mathcal{J}/\mathcal{J}^2$ , viewed as a coherent sheaf on  $S$  via the closed immersion  $\sigma: S \hookrightarrow X$ . Then we have a canonical isomorphism  $\omega_{X/S} \simeq f^* \omega$ .*

*Proof.* By assumption, the torsor  $X_K$  has a  $K$ -rational point; it is then trivial as a torsor under  $J_K$ . According to Lemma 2.1.2, we have then  $d = 1$ . As a result,  $\omega_{X/S} \simeq \mathcal{O}_X$  (Corollary 2.1.4), and the canonical morphism  $f^*f_*\omega_{X/S} \rightarrow \omega_{X/S}$  given by adjunction is an isomorphism. On the other hand, we have the canonical isomorphisms:  $\mathcal{O}_S \simeq (f\sigma)^!\mathcal{O}_S \simeq \sigma^!(\omega_{X/S})[-1] \simeq \sigma^*\omega_{X/S} \otimes \omega^\vee$ . Hence  $\omega \simeq \sigma^*\omega_{X/S} \simeq \sigma^*f^*f_*\omega_{X/S} \simeq f_*\omega_{X/S}$  and we get the following canonical isomorphism  $f^*\omega \simeq f^*f_*\omega_{X/S} \simeq \omega_{X/S}$ , as required.  $\square$

**Remark 2.1.6.** In the general case when  $f$  doesn't necessarily have a section, we can consider the  $S$ -proper minimal regular model  $f': X' \rightarrow S$  of the elliptic curve  $X'_K = \text{Pic}_{X_K/K}^0$ . As a result,  $f'$  has a canonical section given by  $e$ , the schematic closure of the identity element of  $X'_K$  in  $X'$ . Its dualizing sheaf is  $f'^*\omega$  (with  $\omega$  defined by the section  $e$ , see Corollary 2.1.5). One can then recover the sheaf  $\omega_{X/S}$  from the sheaf  $\omega$ , by using some numerical invariants of  $X/S$ ; see §2.3 for more details.

Let  $n \geq 2$  be an integer, and  $\mathcal{L}$  an invertible sheaf on  $X$ , which is of degree 0 on each component of  $X_1$ . Consider the following short exact sequence of sheaves over  $X$

$$0 \rightarrow \mathcal{O}_X(-X_1)|_{X_{n-1}} \rightarrow \mathcal{O}_{X_n} \rightarrow \mathcal{O}_{X_1} \rightarrow 0;$$

by tensoring by the invertible sheaf  $\mathcal{L}^\vee \otimes \omega_{X/S}(X_n)$ , and using the fact that  $X_n = nD$  as divisor on  $X$ , we get the following exact sequence of sheaves over  $X$

$$(13) \quad 0 \rightarrow \mathcal{L}^\vee \otimes \omega_{n-1} \rightarrow \mathcal{L}^\vee \otimes \omega_n \rightarrow \mathcal{L}^\vee \otimes \omega_n|_{X_1} \rightarrow 0,$$

where we also write  $\omega_i$  for the direct image of  $\omega_i$  under the closed immersion  $X_i \hookrightarrow X$ . Hence we have the following exact sequence

$$(14) \quad 0 \rightarrow H^0(X_{n-1}, \mathcal{L}^\vee \otimes \omega_{n-1}) \rightarrow H^0(X_n, \mathcal{L}^\vee \otimes \omega_n) \rightarrow H^0(X_1, \mathcal{L}^\vee \otimes \omega_n|_{X_1}).$$

As a result, we have (by applying Lemma 2.0.2)

$$h^0(X_{n-1}, \mathcal{L}^\vee \otimes \omega_{n-1}) \leq h^0(X_n, \mathcal{L}^\vee \otimes \omega_n) \leq h^0(X_{n-1}, \mathcal{L}^\vee \otimes \omega_{n-1}) + 1.$$

**Lemma 2.1.7.** *With the same notations as above, we have either  $\omega_n \simeq \mathcal{L}|_{X_n}$ , and in this case  $H^0(X_1, \mathcal{L}^\vee \otimes \omega_n|_{X_1}) \simeq k$  and the sequence (14) is right exact, or  $\omega_n \not\simeq \mathcal{L}|_{X_n}$ , in which case the canonical morphism*

$$H^0(X_{n-1}, \mathcal{L}^\vee \otimes \omega_{n-1}) \xrightarrow{\sim} H^0(X_n, \mathcal{L}^\vee \otimes \omega_n)$$

*is bijective.*

*Proof.* Suppose first that  $\omega_n \simeq \mathcal{L}|_{X_n}$ , so that  $\mathcal{L}^\vee \otimes \omega_n \simeq \mathcal{O}_{X_n}$ . Then the last map of the sequence (14) can be identified (in a non canonical way) with the canonical map

$$(15) \quad H^0(X_n, \mathcal{O}_{X_n}) \rightarrow H^0(X_1, \mathcal{O}_{X_1}).$$

By Lemma 2.0.2, we have  $H^0(X_1, \mathcal{O}_{X_1}) \simeq k$ . Hence, every element of  $H^0(X_1, \mathcal{O}_{X_1})$  can be lifted to an element of  $H^0(X_n, \mathcal{O}_{X_n})$ . As a result, the map (15) is surjective and hence the complex (14) is right exact. Thus we obtain the first assertion. Next, suppose  $\omega_n \not\simeq \mathcal{L}|_{X_n}$ . In this case,  $(\omega_n \otimes \mathcal{L}^\vee)|_{X_1} \not\simeq \mathcal{O}_{X_1}$  (use (13) repeatedly) and by Lemma 2.0.2 the term on the right in (14) is trivial. So the canonical map

$$H^0(X_{n-1}, \mathcal{L}^\vee \otimes \omega_{n-1}) \xrightarrow{\sim} H^0(X_n, \mathcal{L}^\vee \otimes \omega_n)$$

is bijective.  $\square$

Next we investigate the Picard group of  $X_n$ . Let  $n \geq 2$  be an integer. The kernel of the surjective morphism  $\text{Pic}(X_n) \rightarrow \text{Pic}(X_{n-1})$  is an  $\mathcal{O}_K$ -module of finite length killed by  $p$ . More precisely, consider the closed immersion  $X_{n-1} \hookrightarrow X_n$ , given by the ideal sheaf  $\mathfrak{N} := \mathcal{I}^{n-1}/\mathcal{I}^n \subset \mathcal{O}_{X_n}$ . The sheaf  $\mathfrak{N}$  is nilpotent (in fact,  $\mathfrak{N}^2 = 0$ ), so we get a short exact sequence of abelian sheaves over  $X_n$  (where we omit to write the obvious direct images):

$$0 \rightarrow 1 + \mathfrak{N} \rightarrow \mathcal{O}_{X_n}^* \rightarrow \mathcal{O}_{X_{n-1}}^* \rightarrow 0.$$

Since  $\mathfrak{N}^2 = 0$ , the morphism  $x \mapsto 1 + x$  defines an isomorphism of abelian sheaves

$$\beta: \mathfrak{N} \rightarrow 1 + \mathfrak{N}.$$

Since  $X_n$  has dimension 1, the cohomology groups  $H^2(X_n, 1 + \mathfrak{N}) \simeq H^2(X_n, \mathfrak{N})$  are zero. In this way we get the following long exact sequence

$$H^0(X_{n-1}, \mathcal{O}_{X_{n-1}}^*) \xrightarrow{\partial^*} H^1(X_n, 1 + \mathfrak{N}) \rightarrow \text{Pic}(X_n) \xrightarrow{\alpha} \text{Pic}(X_{n-1}) \rightarrow 0.$$

On the other hand, from the following exact sequence of sheaves over  $X$ :

$$0 \rightarrow \mathfrak{N} \rightarrow \mathcal{O}_{X_n} \rightarrow \mathcal{O}_{X_{n-1}} \rightarrow 0,$$

we obtain a long exact sequence (recall that  $H^2(X_n, \mathfrak{N}) = 0$ ):

$$H^0(X_{n-1}, \mathcal{O}_{X_{n-1}}) \xrightarrow{\partial} H^1(X_n, \mathfrak{N}) \rightarrow H^1(X_n, \mathcal{O}_{X_n}) \xrightarrow{\alpha'} H^1(X_{n-1}, \mathcal{O}_{X_{n-1}}) \rightarrow 0.$$

We have then the following result:

**Lemma 2.1.8** (Dévissage d'Oort, [15], Proposition in § 6). *Keeping notations as above, one then has  $\beta(\text{im}(\partial)) = \text{im}(\partial^*)$ .*

As a result,  $\ker(\alpha) \simeq \text{coker}(\partial)$  (as abelian sheaves). Since  $\mathfrak{N} = \mathcal{I}^{n-1}/\mathcal{I}^n$  is an  $\mathcal{O}_K$ -module killed by  $p$ , it follows that  $\ker(\alpha) \simeq \text{coker}(\partial)$  is an  $\mathcal{O}_K$ -module of finite length killed by  $p$ . Hence, if we denote by  $d'$  the order of the invertible sheaf  $\mathcal{I}|_{X_1}$ , then for any  $n \geq 2$ , the order of  $\mathcal{I}|_{X_n}$  is of the form  $d'p^\ell$ . Moreover, since the kernel of  $\text{Pic}(X_n) \rightarrow \text{Pic}(X_{n-1})$  is killed by  $p$ , we find that the order of  $\mathcal{I}|_{X_n}$  is equal either to the order of  $\mathcal{I}|_{X_{n-1}}$ , or  $p$  times the order of  $\mathcal{I}|_{X_{n-1}}$ . On the other hand, since  $\mathcal{I}|_{X_K} \simeq \mathcal{O}_{X_K}$ , and since the invertible sheaf  $\mathcal{I}$  is of order  $d$ , by Lemma 6.4.4 in [16], for  $m \in \mathbf{Z}$  sufficiently large, the invertible sheaf  $\mathcal{I}|_{X_m}$  is of order divisible by  $d$ . Moreover, since  $\mathcal{I}$  is of order  $d$ , the order of  $\mathcal{I}|_{X_m}$  divides  $d$ . Hence for  $m \gg 0$ , the order of  $\mathcal{I}|_{X_m}$  is equal to  $d$ . Now, we write  $d = d'p^r$  with  $r \geq 0$  a suitable integer, and for  $i = 0, \dots, r$ , we let  $m_i$  be the smallest integer  $n$  such that  $\mathcal{I}|_{X_n}$  is of order  $d'p^i$ . Let also

$$\phi(n) := h^1(X_n, \mathcal{O}_{X_n})$$

be the length of the  $\mathcal{O}_K$ -module  $H^1(X_n, \mathcal{O}_{X_n})$ . According to Lemma 2.1.7, we have  $\phi(n) \geq \phi(n-1)$ . Moreover,  $\phi(n) > \phi(n-1)$  if and only if  $\omega_n \simeq \mathcal{O}_{X_n}$ , in which case  $\phi(n) = \phi(n-1) + 1$ . One then gets from Lemma 2.1.8 the following Corollary:

**Corollary 2.1.9.** *Let  $n \geq 2$  be an integer. We have either  $\phi(n) = \phi(n-1)$ , in which case the morphism  $\alpha: \text{Pic}(X_n) \rightarrow \text{Pic}(X_{n-1})$  is an isomorphism, or  $\phi(n) = \phi(n-1) + 1$ , in which case  $\ker(\alpha)$  is an  $\mathcal{O}_K$ -module of length 1, and hence a  $k$ -vector space of dimension 1.*

**Lemma 2.1.10.** *Keeping the above notations, one has:*

(i) For  $i = 0, \dots, r$ , the sheaf  $\omega_{m_i}$  is isomorphic to  $\mathcal{O}_{X_{m_i}}$ .

(ii) There exists an integer  $k_i > 0$  such that  $m_{i+1} = m_i + k_i d' p^i$ .

(iii) The integers  $m \in (m_i, m_{i+1}]$  such that  $\phi(m) = \phi(m-1) + 1$  are exactly those which can be written as  $m = m_i + h d' p^i$  for some integer  $h$ .

*Proof.* (i) Let  $n > 1$  be an integer such that the order of  $\mathcal{I}|_{X_n}$  is different from that of  $\mathcal{I}|_{X_{n-1}}$ . We then have  $\phi(n) = \phi(n-1) + 1$ , and the canonical map  $\text{Pic}(X_n) \rightarrow \text{Pic}(X_{n-1})$  has a kernel of length 1 (Corollary 2.1.9). Now use the exact sequence (14) once again. By duality, the injective morphism  $H^0(X_{n-1}, \omega_{n-1}) \rightarrow H^0(X_n, \omega_n)$  has a non trivial cokernel, and this implies that  $\omega_n \simeq \mathcal{O}_{X_n}$  (Lemma 2.1.7).

For the assertions (ii) and (iii), recall that  $\omega_{m_i} = \omega_{X/S}(m_i D)|_{X_{m_i}}$ , and by Corollary 2.1.4, there exists an integer  $n$  ( $1 \leq n \leq d-1$ ) such that  $\omega_{X/S} \simeq \mathcal{I}^n$ . Let  $\mathcal{L}_m := \omega_{X/S}(mD)$ , then  $\mathcal{L}_m \simeq \mathcal{I}^{n-m}$ . But  $\omega_{m_{i+1}} = \mathcal{L}_{m_{i+1}}|_{X_{m_{i+1}}} = \mathcal{I}^{n-m_{i+1}}|_{X_{m_{i+1}}} \simeq \mathcal{O}_{X_{m_{i+1}}}$ . Hence,  $\omega_{m_{i+1}}|_{X_{m_i}} = \mathcal{I}^{n-m_{i+1}}|_{X_{m_i}} \simeq \mathcal{O}_{X_{m_i}}$  by (i). Since  $\omega_{m_i} = \mathcal{I}^{n-m_i}|_{X_{m_i}} \simeq \mathcal{O}_{X_{m_i}}$ , we have  $\mathcal{I}^{m_{i+1}-m_i}|_{X_{m_i}} \simeq \mathcal{O}_{X_{m_i}}$ , and thus there exists an integer  $k_i > 0$  such that  $m_{i+1} = m_i + k_i d' p^i$ . The same argument also gives us that, for an integer  $m$  so that  $m_{i+1} \geq m > m_i$  and  $\phi(m) > \phi(m-1)$ , there exists an integer  $0 < h \leq k_i$  verifying  $m = m_i + h d' p^i$ . Conversely, let  $m$  be an integer of the form  $m = m_i + h d' p^i$  (for some  $0 < h \leq k_i$ ; we show that  $\phi(m) > \phi(m-1)$ ). We may assume that  $m < m_{i+1}$ , hence  $\mathcal{I}|_{X_m}$  is of order  $d' p^i$ . By Lemma 2.1.7, we only need to show that  $\omega_m \simeq \mathcal{O}_{X_m}$ . But

$$\begin{aligned} \omega_m &\simeq \mathcal{I}^{n-m}|_{X_m} = \mathcal{I}^{n-m_i-hd'p^i}|_{X_m} = \mathcal{I}^{n-m_{i+1}+m_{i+1}-m_i-hd'p^i}|_{X_m} \\ &\simeq \omega_{m_{i+1}}|_{X_m} \otimes \mathcal{I}^{(k_i-h)d'p^i}|_{X_m} \simeq \mathcal{O}_{X_m} \end{aligned}$$

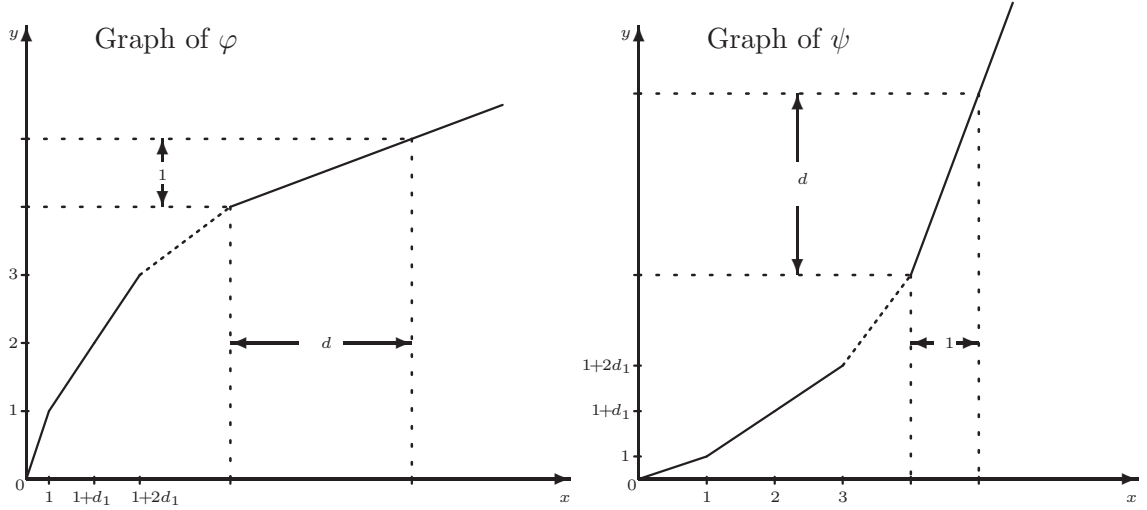
since  $\omega_{m_{i+1}} \simeq \mathcal{O}_{X_{m_{i+1}}}$  and  $\mathcal{I}|_{X_m}$  is of order  $d' p^i$ . This completes the proof.  $\square$

## 2.2 The function $\psi$

We now come to the key construction of this section. We define a function  $\varphi: \mathbf{R}_{\geq 0} \rightarrow \mathbf{R}_{\geq 0}$  such that the graph of  $\varphi$  is just the concave envelope of the set  $\{(n, \phi(n)) \mid n \in \mathbf{Z}_{\geq 0}\} \subset \mathbf{R}^2$ . Then  $\varphi$  is a continuous function, strictly increasing, and piecewise linear. Moreover,  $\varphi(0) = 0$ , and  $\varphi(1) = 1$ . Let  $\psi: \mathbf{R}_{\geq 0} \rightarrow \mathbf{R}_{\geq 0}$  be its inverse. Hence,  $\psi$  is also continuous and piecewise linear. For any integer  $n \geq 1$ ,  $\psi(n)$  is just the smallest integer  $m \geq 1$  such that  $\phi(m) = n$ . If we denote by  $d_n$  the order of the invertible sheaf  $\mathcal{I}|_{X_{\psi(n)}}$ , then by Lemma 2.1.10 we have

$$(16) \quad \psi(n+1) = \psi(n) + d_n,$$

and for all  $m \in \mathbf{Z}$  such that  $\psi(n) \leq m < \psi(n+1)$ , the morphism of groups  $\text{Pic}(X_m) \rightarrow \text{Pic}(X_{\psi(n)})$  is an isomorphism (Corollary 2.1.9). We call the functions  $\varphi, \psi: \mathbf{Z}_{\geq 1} \rightarrow \mathbf{Z}_{\geq 1}$  the Herbrand functions, which are exact analogues of the Herbrand functions used by Serre in [19].



To finish this section, we present some corollaries of the previous discussion, which will be useful in the next section. The first follows directly from Lemma 2.1.7 by duality.

**Corollary 2.2.1.** *Let  $\mathcal{L}$  be an invertible sheaf on  $X$ , of degree 0 on each component of  $X_1$ , and let  $n$  be an integer  $\geq 2$ . Then we have  $h^1(X_{n-1}, \mathcal{L}|_{X_{n-1}}) \leq h^1(X_n, \mathcal{L}|_{X_n}) \leq h^1(X_{n-1}, \mathcal{L}|_{X_{n-1}}) + 1$ . Moreover,  $h^1(X_n, \mathcal{L}|_{X_n}) = h^1(X_{n-1}, \mathcal{L}|_{X_{n-1}}) + 1$  if and only if  $\mathcal{L}|_{X_n} \simeq \omega_n$ .*

**Corollary 2.2.2.** *Let  $\mathcal{L}$  be an invertible sheaf on  $X$ , of degree 0 on each component of  $X_1$ , and let  $n$  be an integer  $\geq 1$ . Then, if the morphism  $H^1(X, \mathcal{L}) \rightarrow H^1(X_n, \mathcal{L}|_{X_n})$  is not bijective, there exists an integer  $m > n$  such that  $\mathcal{L}^\vee|_{X_m} \otimes \omega_m$  is trivial on  $X_m$ .*

*Proof.* We first remark that the morphism  $H^1(X, \mathcal{L}) \rightarrow H^1(X_n, \mathcal{L}|_{X_n})$  is surjective, and we have  $H^1(X, \mathcal{L}) = \varprojlim_{m \geq 0} H^1(X_m, \mathcal{L}|_{X_m})$ . Moreover, the morphism is not bijective if and only if there exists  $m > n$  such that  $H^1(X_m, \mathcal{L}|_{X_m}) \rightarrow H^1(X_{m-1}, \mathcal{L}|_{X_{m-1}})$  is not injective. By duality, this is equivalent to saying that the injective morphism  $H^0(X_{m-1}, \mathcal{L}^\vee|_{X_{m-1}} \otimes \omega_{m-1}) \rightarrow H^0(X_m, \mathcal{L}^\vee|_{X_m} \otimes \omega_m)$  is not surjective. Hence, we have  $\mathcal{L}^\vee|_{X_m} \otimes \omega_m \simeq \mathcal{O}_{X_m}$  (Lemma 2.1.7) and one concludes.  $\square$

For  $\mathcal{L}$  an invertible sheaf on  $X$  of degree 0 on each component of  $X_1$ , the  $\mathcal{O}_K$ -module  $H^1(X, \mathcal{L})$  is of *infinite* length if and only if  $\mathcal{I}|_{X_K} \simeq \mathcal{O}_{X_K}$ , that is, if and only if  $\mathcal{L}$  is isomorphic to a power of  $\mathcal{I} = \mathcal{O}_X(-D)$  (see the proof of Corollary 2.1.4). Hence when  $\mathcal{L}|_{X_K}$  is not isomorphic to  $\mathcal{O}_{X_K}$ , the  $\mathcal{O}_K$ -module  $H^1(X, \mathcal{L})$  is of *finite* length. Moreover, we have

**Corollary 2.2.3.** *Let  $\mathcal{L}$  be an invertible sheaf on  $X$  of degree 0 on each component of  $X_1$ , and let  $n \geq 1$  be an integer. Suppose that the  $\mathcal{O}_K$ -module  $H^1(X, \mathcal{L})$  is of length  $\geq n$ , then  $\mathcal{L}|_{X_{\psi(n)}} \simeq \mathcal{I}^i|_{X_{\psi(n)}}$  with  $i$  a suitable integer.*

*Proof.* We will prove by induction that, under our assumptions and for any  $n'$ ,  $1 \leq n' \leq n-1$ , the  $\mathcal{O}_K$ -module  $H^1(X_{\psi(n'+1)-1}, \mathcal{L}|_{X_{\psi(n'+1)-1}})$  is of length  $n'$ . As a result, the canonical map  $H^1(X, \mathcal{L}) \rightarrow H^1(X_{\psi(n)-1}, \mathcal{L}|_{X_{\psi(n)-1}})$  is not bijective (here if  $n = 1$ , we define by convention that  $H^1(X_0, \mathcal{L}|_{X_0}) = 0$ ). Hence the Corollary 2.2.2 provides an integer  $m > \psi(n) - 1$  such that  $\mathcal{L}|_{X_m} \simeq \omega_m \cong \mathcal{I}|_{X_m}^{\bar{n}-m}$ , where  $\bar{n}$  is the integer appearing in Corollary 2.1.4. Since  $\psi(n) \leq m$ , the Corollary follows.

We begin with the case where  $n' = 1$  (hence  $n \geq 2$ ). By Lemma 2.0.2, either the  $\mathcal{O}_K$ -module  $H^1(X_1, \mathcal{L}|_{X_1})$  is of length 1, which is equivalent to saying that  $\mathcal{L}|_{X_1} \simeq \mathcal{O}_{X_1}$ , or  $H^1(X_1, \mathcal{L}|_{X_1})$  is trivial, and in this case, the natural morphism

$$H^1(X, \mathcal{L}) \rightarrow H^1(X_1, \mathcal{L}|_{X_1})$$

is not bijective. By Corollary 2.2.2 there is then an integer  $m > 1$ , such that  $\mathcal{L}|_{X_m} \simeq \omega_m = \omega_{X/S}(X_m)|_{X_m}$ . Hence  $\mathcal{L}|_{X_1} \simeq \omega_m|_{X_1}$  is a power of  $\mathcal{I}|_{X_1}$  (Corollary 2.1.4). Thus, in both cases, we have  $\mathcal{L}|_{X_1} \simeq \mathcal{I}|_{X_1}^i$  for some suitable integer  $i$ . Moreover, for  $m$  an integer such that  $1 \leq m \leq \psi(2) - 1$ , the canonical morphism  $\text{Pic}^0(X_m) \rightarrow \text{Pic}^0(X_1)$  is bijective (see Corollary 2.1.9), hence  $\mathcal{L}|_{X_m} \simeq \mathcal{I}|_{X_m}^i$ . But  $\psi(2) = \psi(1) + d_1 = 1 + d_1$ , and by definition, there exists a unique integer  $m$  such that  $1 \leq m \leq \psi(2) - 1$ , and  $\mathcal{L}|_{X_m} \simeq \omega_m$ . Hence, by Lemma 2.2.1, we find that the  $\mathcal{O}_K$ -module  $H^1(X_{\psi(2)-1}, \mathcal{L}|_{X_{\psi(2)-1}})$  is of length 1. Suppose now that the above assertion has been verified for  $1 \leq n' - 1 < n$  (with  $n' < n$ ). Under the assumptions of the Lemma, the map

$$H^1(X, \mathcal{L}) \rightarrow H^1(X_{\psi(n')-1}, \mathcal{L}|_{X_{\psi(n')-1}})$$

is not surjective. Hence, there exists an integer  $m \geq \psi(n')$ , such that  $\mathcal{L}|_{X_m} \simeq \omega_m$ , and so  $\mathcal{L}|_{X_{\psi(n')}} \simeq \mathcal{I}|_{X_{\psi(n')}}^i$  for  $0 \leq i < d_{n'}$ . On the other hand, since  $\psi(n' + 1) = \psi(n') + d_{n'}$  (see (16)), there exists a unique integer  $m$  such that  $\psi(n') \leq m \leq \psi(n' + 1) - 1$ , and  $\mathcal{L}|_{X_m} \simeq \omega_m$ , in particular, the  $\mathcal{O}_K$ -module  $H^1(X_{\psi(n'+1)-1}, \mathcal{L}|_{X_{\psi(n'+1)-1}})$  is of length  $n'$ . This finishes the induction, and hence also the proof of the Corollary.  $\square$

## 2.3 Numerical studies

We maintain the notation of Remark 2.1.6. In particular,  $f': X' \rightarrow S$  is the proper minimal regular  $S$ -model of  $X'_K = \text{Pic}_{X_K/K}^0$ . According to [11], Theorem 3.8, there exists a morphism of  $\mathcal{O}_K$ -modules

$$\tau_X: H^1(X, \mathcal{O}_X) \rightarrow H^1(X', \mathcal{O}_{X'})$$

which extends the natural isomorphism over the generic fibre. Moreover, its kernel is the torsion part of  $H^1(X, \mathcal{O}_X)$  ([11], 3.1 a)), and the  $\mathcal{O}_K$ -modules  $\ker(\tau_X)$  and  $\text{coker}(\tau_X)$  have the same length. In this section we give an estimate for this length.

By duality, we obtain the following map

$$\tau_X^\vee: H^0(X', \omega_{X'/S}) \simeq (H^1(X', \mathcal{O}_{X'}))^\vee \rightarrow (H^1(X, \mathcal{O}_X))^\vee \simeq H^0(X, \omega_{X/S}).$$

On the other hand, there is a canonical isomorphisms  $f'_* \omega_{X'/S} \simeq f'_* f'^* \omega \simeq \omega$  (see Corollary 2.1.5 with  $\omega$  defined via the  $S$ -section of  $f'$  associated with the identity element of  $X'_K$ ). Hence we get following canonical map:

$$\tau_X^\vee: \omega(S) \simeq H^0(X', \omega_{X'/S}) \rightarrow H^0(X, \omega_{X/S}) = f_* \omega_{X/S}(S)$$

which is injective, but not an isomorphism in general. Since  $S$  is affine,  $\tau_X^\vee$  also corresponds to an injective morphism of sheaves, again denoted by  $\tau_X^\vee$ :

$$(17) \quad \tau_X^\vee: \omega \rightarrow f_* \omega_{X/S}.$$

which gives by adjunction the following non zero canonical morphism of sheaves on  $X$ :

$$\tau_X^{\vee'}: f^* \omega \rightarrow \omega_{X/S}.$$

Since  $\tau_X^\vee$  is an isomorphism on the generic fibre  $X_K$  of  $X$ , the same holds for  $\tau_X^{\vee'}$ . Under the identification given by the restriction of  $\tau_X^{\vee'}$  to  $X_K$ ,  $f^*(\omega)$  and  $\omega_{X/S}$  are naturally identified with two  $\mathcal{O}_X$ -submodules of  $\omega_{X_K/K} := (\omega_{X/S})|_{X_K}$ .



**Lemma 2.3.1.** *We have  $\omega_{X/S} = f^*\omega \otimes \mathcal{I}^{-\chi}$  for a suitable non negative integer  $\chi$ , as submodules of  $\omega_{X_K/K}$ . Moreover,  $[\chi/d]$  is the length of the torsion part of  $H^1(X, \mathcal{O}_X)$ .*

*Proof.* Since  $f^*\omega$  is invertible and the scheme  $X$  is regular, the generically injective morphism  $\tau_X^{\vee'}$  is automatically injective. By tensoring both sides with  $\omega_{X/S}^{-1}$ , we get an invertible ideal sheaf

$$\mathcal{J} := f^*\omega \otimes \omega_{X/S}^{-1} \hookrightarrow \mathcal{O}_X.$$

The closed subscheme  $V(\mathcal{J})$  of  $X$  defined by  $\mathcal{J}$  has support contained in  $X_s$ . Moreover, the intersection numbers  $V(\mathcal{J}) \cdot C_i = 0$  for any irreducible component  $C_i$  of  $X_s$  (Lemma 2.1.3). Hence the effective divisor  $V(\mathcal{J}) \hookrightarrow X$  is a multiple of  $D = X_1 = V(\mathcal{I}) \hookrightarrow X$ . So one can find some non negative integer  $\chi \in \mathbb{N}$  such that  $\mathcal{J} = f^*\omega \otimes \omega_{X/S}^{-1} = \mathcal{I}^\chi$ . In other words, we find the following identification  $\omega_{X/S} = f^*\omega \otimes \mathcal{I}^{-\chi}$  as submodules of  $\omega_{X_K/K}$ . This proves the first assertion. Under the latter identification, the morphism  $\tau_X^{\vee'}$  is then obtained from the canonical map:  $\mathcal{O}_X \hookrightarrow \mathcal{I}^{-\chi}$  after tensoring both sides by  $f^*\omega$ . Hence the morphism  $\tau_X^\vee$  in (17) can now be described as the following composition:

$$\tau_X^\vee : \omega \simeq f_*f^*\omega = f_*(f^*\omega \otimes \mathcal{O}_X) \rightarrow f_*(f^*\omega \otimes \mathcal{I}^{-\chi}) = f_*\omega_{X/S},$$

where the first isomorphism is just the adjunction map, and the third is induced by the canonical injection  $\mathcal{O}_X \hookrightarrow \mathcal{I}^{-\chi}$ . Hence, under the canonical identification, by using the projection formula

$$f_*(f^*\omega \otimes \mathcal{I}^{-\chi}) \simeq \omega \otimes f_*(\mathcal{I}^{-\chi}),$$

the morphism  $\tau_X^\vee$  is obtained from the canonical map  $\mathcal{O}_S = f_*\mathcal{O}_X \rightarrow f_*(\mathcal{I}^{-\chi})$  after tensoring by the invertible sheaf  $\omega$ . Now if we identify these two sheaves as  $\mathcal{O}_S$ -submodule of  $f_{K,*}\mathcal{O}_{X_K} = \mathcal{O}_{\text{Spec}(K)}$  (here  $f_K$  is the generic fibre of  $f$ ), we have  $f_*(\mathcal{I}^{-\chi}) = \pi^{-[\chi/d]}\mathcal{O}_S \subset \mathcal{O}_{\text{Spec}(K)}$ . Indeed, if we write  $f_*(\mathcal{I}^{-\chi}) = \pi^{-r}\mathcal{O}_S$  for some non negative integer  $r$ , then  $r$  is the largest integer  $r'$  such that  $\pi^{-r'}\mathcal{O}_S \subset f_*(\mathcal{I}^{-\chi})$ . But this last inclusion is equivalent to the inclusion

$$f^*(\pi^{-r'}\mathcal{O}_S) = \mathcal{I}^{-dr'} \subset \mathcal{I}^{-\chi},$$

hence is also equivalent to the condition  $-dr' \geq -\chi$ , namely  $r' \leq \chi/d$ . The maximum of the possible  $r'$  is then given by  $r = [\chi/d]$ , and thus we obtain  $f_*(\mathcal{I}^{-\chi}) = \pi^{-[\chi/d]}\mathcal{O}_S$ . Hence, if we identify  $\omega$  and  $f_*\omega_{X/S}$  as  $\mathcal{O}_S$ -submodule of  $\omega_K = \omega \otimes_{\mathcal{O}_S} \mathcal{O}_{\text{Spec}(K)}$ , the injection (17) gives us the following equality inside  $\omega_K$ :  $f_*(\omega_{X/S}) = \pi^{-[\chi/d]}\omega$ . As a result, we find that  $\text{coker}(\tau_X^\vee)$  and the torsion part of  $H^1(X, \mathcal{O}_X)$  are both of length  $[\chi/d]$ .  $\square$

**Proposition 2.3.2.** *In the foregoing notation, one has*

$$(18) \quad \chi = d[(1 - 1/d) + k_0(1 - 1/p^r) + \cdots + k_{r-1}(1 - 1/p)],$$

where the  $k_i$  are the integers introduced in Lemma 2.1.10.

*Proof.* By Lemma 2.1.10, we have  $\phi(m_r) = 1 + k_0 + \cdots + k_{r-1}$ , and for  $n > m_r$ , we have  $\phi(n) = \phi(n-1) + 1$  if and only if  $n - m_r$  is a multiple of  $d = d'p^r$ . In particular, if we write  $m_r = hd - a$  for non negative integers  $h$ ,  $0 \leq a < d$ , we have  $\phi(m_r) = \phi(hd)$ . Let  $\mathcal{M} = R^1f_*\mathcal{O}_X$ , and  $\mathcal{T} \subset \mathcal{M}$  be the torsion subsheaf of  $\mathcal{M}$ . Consider  $\mathcal{L} = \mathcal{M}/\mathcal{T}$ , which is free of rank 1 over  $S$ . We have  $R^1f_*\mathcal{O}_{X_{nd}} = R^1f_*(\mathcal{O}_X/\pi^n\mathcal{O}_X) \simeq R^1f_*(\mathcal{O}_{X_{nd}}) = \mathcal{M}/\pi^n\mathcal{M}$ . Hence, for  $n \geq h$ , the length of  $\mathcal{M}/\pi^n\mathcal{M}$ , i.e.,  $\phi(nd)$ , increases by 1 with  $n$ . This means that  $\mathcal{T}$  is killed by  $\pi^h$ , and that  $\ell(\mathcal{M}/\pi^n\mathcal{M}) = \ell(\mathcal{T}) + \ell(\mathcal{L}/\pi^n\mathcal{L})$ . In particular, if we take  $n = h$ , we get

$$(19) \quad \phi(m_r) = \phi(hd) = \ell(R^1f_*\mathcal{O}_{X_{hd}}) = \ell(\mathcal{M}/\pi^h\mathcal{M}) = \ell(\mathcal{T}) + h.$$

On the other hand,  $\omega_{m_r} = \mathcal{I}^{-(\chi+m_r)}|_{X_{m_r}}$  is the trivial invertible sheaf, and since  $\mathcal{I}|_{X_{m_r}}$  is of order  $d$ , there exists an integer  $\alpha$  such that  $\chi + m_r = \alpha d$ . Hence  $\chi = (\alpha - h)d + a$ , and we have  $\ell(T) = [\chi/d] = \alpha - h$ . Thus, (by using the equality (19) and Lemma 2.3.1), we find that  $\ell(T) = [\chi/d] = \alpha - h = \phi(m_r) - h$ . Hence  $\alpha = \phi(m_r)$ , and

$$\begin{aligned}\chi &= \phi(m_r)d - m_r \\ &= (1 + k_0 + \cdots + k_{r-1})d - (1 + k_0d' + \cdots + k_{r-1}d'p^{r-1}) \\ &= d[(1 - 1/d) + k_0(1 - 1/p^r) + \cdots + k_{r-1}(1 - 1/p)].\end{aligned}$$

□

**Corollary 2.3.3.** *The following conditions are equivalent:*

- (i)  $X/S$  is cohomologically flat (in dimension 0).
- (ii)  $\chi < d$ .
- (iii)  $r = 0$ .
- (iv)  $\mathcal{I}|_{X_1}$  is of order  $d$ .

Moreover, if these conditions are satisfied, we have  $\chi = d - 1$ .

*Proof.* The  $S$ -scheme  $X$  is cohomologically flat if and only if  $\mathcal{T}$ , the torsion subsheaf of  $R^1f_*\mathcal{O}_X$ , is trivial, *i.e.*, if and only if  $\ell(\mathcal{T}) = [\chi/d] = 0$ ; the latter assertion is equivalent to saying that  $\chi < d$ , hence (i)  $\Leftrightarrow$  (ii). The equivalence between (iii) and (iv) comes from the definition of  $r$ . On the other hand, if  $r > 0$ , then by definition of the integers  $k_i$ , we must have  $k_i > 0$  for  $i = 0, 1, \dots, r-1$ . Hence by (18),  $\chi < d$  if and only if  $r = 0$ . This gives the equivalence between (ii) and (iii), which completes the proof. □

**Remark 2.3.4.** 1. Once the above equivalent conditions are verified, we say that the torsor  $X_K$  is *tamely ramified*; otherwise, we say that the torsor  $X_K$  is *wildly ramified*. Hence, by Lemma 2.1.8, if  $(d, p) = 1$ , the torsor  $X_K$  is automatically tame. The converse is not true in general (cf. [16], Remark 9.4.3 d)).

- 2. We refer to Corollary 3.4.5 for more equivalent descriptions about the tameness by using the Picard functors.

### 3 Filtrations and comparison of the pro-algebraic structures

Let  $n \geq 1$  be an integer. We have a canonical morphism of groups  $\text{Pic}^0(X) \rightarrow \text{Pic}^0(X_n)$ . In this way we obtain a filtration on the group of  $S$ -points of the Picard functor  $\text{Pic}_{X/S}^0(S) = \text{Pic}^0(X)$ . On the other hand, the group  $J(S)$  of the  $S$ -points of the identity component  $J$  of the Néron model of the elliptic curve  $\text{Pic}_{X_K/K}^0$  is naturally filtered by the powers of  $\pi$  (*i.e.*, the filtration given by the canonical morphism of groups  $J(S) \rightarrow J(S_n)$  where  $S_n = \text{Spec}(\mathcal{O}_K/\pi^n\mathcal{O}_K)$ ). The reader should bear in mind that  $X_n$  denotes the closed subscheme of  $X$  defined by the ideal sheaf  $\mathcal{I}$  and that  $\mathcal{I}^d = \pi\mathcal{O}_X$ . The aim of this section is to study the relation between these two filtrations with respect to the natural morphism of sheaves  $q: \text{Pic}_{X/S}^0 \rightarrow J$  in (10). The result can be stated in a satisfying form by using the Greenberg realization functors (see Theorem 3.4.3).

For this construction, we shall make intensive use of the notion of *dilatation* of a group scheme. We recall this construction briefly (see [5], § 3.2, for more details). Let  $H$  be a smooth

$S$ -group scheme of finite type,  $W \hookrightarrow H_s$  a smooth subgroup scheme over  $k$ . Denote by  $\mathcal{J}$  the ideal sheaf of  $W \hookrightarrow H$ . Let us denote by  $\text{Bl}_W(H)$  the blowing-up of  $H$  along the center  $W \hookrightarrow H$ . Then, by definition, *the dilatation of  $H$  along the center  $W \hookrightarrow H$*  is the largest open scheme  $H' \subset \text{Bl}_W(H)$  such that the ideal  $\mathcal{J}\mathcal{O}_{H'} \subset \mathcal{O}_{H'}$  is generated by  $\pi$ . According to [5], 3.2/3,  $H'$  is a smooth  $S$ -group scheme, satisfying the following universal property: let  $Z$  be a flat  $S$ -scheme and  $v: Z \rightarrow H$  a morphism of  $S$ -schemes such that its restriction to special fibres,  $v_s: Z_s \rightarrow H_s$ , factors through  $W \hookrightarrow H_s$ , then there exists a unique  $S$ -morphism  $v': Z \rightarrow H'$  rendering the obvious diagram commutative.

In order to simplify the presentation, for the rest of the section we will use the following notations: let  $n \leq m$  be two non-negative integers, possibly  $m = \infty$ , and denote by  $P^{[n,m]}$  the kernel of the canonical morphism of functors  $\text{Pic}_{X_m/S}^0 \rightarrow \text{Pic}_{X_n/S}^0$ . Here, we set  $X_\infty = X$  and  $\text{Pic}_{X_0/S}^0 = \{0\}$ , the final object in the category of abelian fppf-sheaves on  $S$ . Furthermore let

$$P^{[n]} := P^{[n,\infty]} = \ker(\text{Pic}_{X/S}^0 \rightarrow \text{Pic}_{X_n/S}^0), \quad P_{[n]} := P^{[0,n]} = \text{Pic}_{X_n/S}^0.$$

In particular,  $P^{[0]} = \text{Pic}_{X/S}^0$ . For any integer  $n \geq 1$ , we define by induction a smooth  $S$ -group scheme  $J^{[n]}$  as the the dilatation of  $J^{[n-1]}$  along the unit element of the special fibre of  $J_s^{[n-1]}$  (here,  $J^{[0]} := J$ ). According to the universal property of dilatations, for any  $n \in \mathbf{Z}_{\geq 0}$ , we have the following exact sequence:

$$(20) \quad 0 \rightarrow J^{[n]}(S) \rightarrow J(S) \rightarrow J(S_n) \rightarrow 0.$$

Hence we get a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & J^{[n]}(S) & \longrightarrow & J(S) & \longrightarrow & J(S_n) \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & J^{[n-1]}(S) & \longrightarrow & J(S) & \longrightarrow & J(S_{n-1}) \longrightarrow 0 \end{array}$$

From this it follows that the canonical morphism of abstract groups

$$\delta : J^{[n-1]}(S_1) \simeq \text{coker}(J^{[n]}(S) \rightarrow J^{[n-1]}(S)) \rightarrow \ker(J(S_n) \rightarrow J(S_{n-1}))$$

is an isomorphism, where for the first identification one applies (20) with  $J^{[n-1]}$  on place of  $J$  and  $n = 1$ . Moreover, from the diagram given above, we obtain the following commutative diagram

$$(21) \quad \begin{array}{ccccccc} 0 & \longrightarrow & J^{[n-1]}(S) & \longrightarrow & J(S) & \longrightarrow & J(S_{n-1}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & J^{[n-1]}(S_1) & \xrightarrow{\delta} & J(S_n) & \longrightarrow & J(S_{n-1}) \longrightarrow 0 \end{array}$$

### 3.1 Pro-algebraic structures

Recall that, in this paper, a *pro-algebraic group over  $k$*  is a pro-object in the category of  $k$ -group schemes of finite type (see § 1.2). The aim of this subsection is to show that, with the help of Greenberg realization functors, the morphism  $q: \text{Pic}^0(X) \rightarrow J(S)$  is pro-algebraic in nature.

Let  $n \geq 1$  be an integer. Consider  $\text{Gr}(P_{[n]})$  the Greenberg realization of the Picard functor  $P_{[n]} = \text{Pic}_{X_n/S}^0$  (§1.2.1). The natural morphism of functors  $P_{[n+1]} \rightarrow P_{[n]}$  induces a morphism

of smooth  $k$ -group schemes  $\alpha_n: \mathrm{Gr}(\mathrm{P}_{[n+1]}) \rightarrow \mathrm{Gr}(\mathrm{P}_{[n]})$ . Thus we obtain a pro-algebraic group over  $k$  (in the sense of § 1.2)

$$\mathrm{Gr}(\mathrm{Pic}_{X/S}^0) := \{(\mathrm{Gr}(\mathrm{P}_{[n]}), \alpha_n)\}_{n \geq 1}.$$

Moreover, in view of the following Lemma, which follows from [10] and Corollary 2.1.9, we know that this pro-algebraic  $k$ -group is pro-smooth.

**Lemma 3.1.1.** *Maintain the previous notation. Then, the morphism  $\alpha_n$  is a smooth and surjective morphism of smooth connected  $k$ -group schemes. Moreover, either  $\alpha_n$  is an isomorphism, in which case, we have  $\phi(n+1) = \phi(n)$ , or  $\ker(\alpha_n)$  is a  $k$ -vector group of dimension 1, in which case we have  $\phi(n+1) = \phi(n) + 1$ .*

On passing to the projective limit of the associated perfect group schemes  $\mathbf{Gr}(\mathrm{P}_{[n]})$  and using the fact that

$$\mathrm{Pic}^0(X) = \varprojlim \mathrm{Pic}^0(X_n) = \varprojlim \mathrm{Gr}(\mathrm{P}_{[n]})(k),$$

we get a pro-algebraic structure in the sense of Serre on the group  $\mathrm{Pic}_{X/S}^0(S) = \mathrm{Pic}^0(X)$ . We will denote the pro-algebraic group so obtained by

$$\mathbf{Pic}^0(\mathbf{X}) := \varprojlim \mathbf{Gr}(\mathrm{P}_{[n]}).$$

Similarly, since  $\mathrm{P}^{[n]}(S) = \ker(\mathrm{Pic}_{X/S}^0(S) \rightarrow \mathrm{P}_{[n]}(S))$ , we find that the group  $\mathrm{P}^{[n]}(S)$  can also be endowed with a pro-algebraic structure in the sense of Serre, denoted by  $\mathbf{P}^{[n]}(\mathbf{S})$ . Thus we obtain a decreasing filtration of  $\mathbf{Pic}^0(\mathbf{X})$  by its sub-pro-algebraic groups:

$$(22) \quad \dots \subset \mathbf{P}^{[n+1]}(\mathbf{S}) \subset \mathbf{P}^{[n]}(\mathbf{S}) \subset \dots \subset \mathbf{P}^{[1]}(\mathbf{S}) \subset \mathbf{P}^{[0]}(\mathbf{S}) = \mathbf{Pic}^0(\mathbf{X}).$$

Secondly, from the  $S$ -group scheme  $J$ , we can construct a pro-smooth pro-algebraic  $k$ -group  $\{(\mathrm{Gr}_n(J), \beta_n)\}_{n \geq 1}$ , and hence a pro-algebraic algebraic group in the sense of Serre

$$\mathbf{J}(\mathbf{S}) := \mathbf{Gr}(J) = \varprojlim \mathbf{Gr}_n(J)$$

whose group of  $k$ -points is  $J(S)$ . Moreover, the canonical map  $J(S) = \mathrm{Gr}(J)(k) \rightarrow \mathrm{Gr}_n(J)(k) = J(S_n)$  is also pro-algebraic in nature, hence its kernel can also be endowed with a pro-algebraic structure. This last pro-algebraic group, according to the short exact sequence (20), is just the sub-pro-algebraic group  $\mathbf{J}^{[n]}(\mathbf{S}) \subset \mathbf{J}(\mathbf{S})$  induced by the canonical map of  $S$ -group schemes  $J^{[n]} \rightarrow J$ . In this way, we also obtain a decreasing filtration of  $\mathbf{J}(\mathbf{S})$  by its sub-pro-algebraic groups:

$$(23) \quad \dots \subset \mathbf{J}^{[n+1]}(\mathbf{S}) \subset \mathbf{J}^{[n]}(\mathbf{S}) \subset \dots \subset \mathbf{J}^{[1]}(\mathbf{S}) \subset \mathbf{J}^{[0]}(\mathbf{S}) = \mathbf{J}(\mathbf{S}).$$

On the other hand, for each integer  $n \geq 1$ , the morphism  $q: \mathrm{Pic}_{X/S}^0 \rightarrow J$  induces a morphism of functors  $\mathrm{Pic}_{X/S}^0 \times_S S_n = \mathrm{P}_{[nd]} \times_S S_n \rightarrow J \times_S S_n$ , hence a morphism of algebraic  $k$ -groups:

$$\mathrm{Gr}(\mathrm{P}_{[nd]}) \rightarrow \mathrm{Gr}_n(J).$$

In particular, we obtain a morphism of pro-algebraic groups:

$$(24) \quad \mathrm{Gr}(\mathrm{Pic}_{X/S}^0) = \{(\mathrm{Gr}(\mathrm{P}_{[n]}), \alpha_n)\}_{n \geq 1} \rightarrow \{(\mathrm{Gr}_n(J), \beta_n)\}_{n \geq 1} = \mathrm{Gr}(J).$$

In this way, we discover that the canonical morphism  $q: \text{Pic}^0(X) \rightarrow J(S)$  is the morphism on  $k$ -rational points induced by a morphism of Serre pro-algebraic groups:

$$(25) \quad q: \text{Pic}^0(X) \rightarrow J(S).$$

In fact, we can be more precise in comparing the two filtrations (22) and (23). The main result of this section (see Theorem 3.4.3) says that the above filtrations are compatible via  $q$  and, this fact suggests that the morphism  $q$  should be thought as an analogue of the norm map studied by Serre in [19]. In order to explore the compatibility between the two filtrations, we start by proving a useful result on the length of torsion sheaves.

### 3.2 A result on intersection theory

The results of this section hold for  $\mathcal{O}_K$  any discrete valuation ring; as usual we denote by  $S$  its spectrum and by  $s$  the closed point. Moreover, for a torsion coherent sheaf defined over the spectrum of a discrete valuation ring, we denote by  $\ell(\mathcal{M})$  the length of  $\mathcal{M}$ .

**Proposition 3.2.1.** *Let  $Z$  be a smooth  $S$ -scheme of finite type with  $\xi$  a generic point of its special fibre  $Z_s$ . Let  $\alpha: S \rightarrow Z$  be a section of  $Z/S$  such that  $x = \alpha(s) \in \overline{\{\xi\}} \subset Z_s$ . Let  $\mathcal{M}$  be a coherent torsion sheaf over  $Z$ , whose support  $\text{Supp}(\mathcal{M})$  is purely of codimension 1 in  $Z$ . Suppose that  $\mathcal{M}$  is of length  $\ell$  at  $\xi$ , and that  $\alpha(S) \not\subset \text{Supp}(\mathcal{M})$ .*

(1) *We have  $\ell(\alpha^*\mathcal{M}) \geq \ell$ , with equality if and only if the following conditions are satisfied: the support of  $\mathcal{M}$  at  $\alpha(s)$  is contained in  $Z_s$  and  $\mathcal{M}$  is Cohen-Macaulay at  $\alpha(s)$ .*

(2) *Suppose furthermore that the support of  $\mathcal{M}_K$  on  $Z_K$  is not empty, and let  $H_K = \text{Supp}(\mathcal{M}_K)_{\text{red}}$  with  $H = \overline{H_K} \subset Z$  its schematic closure in  $Z$  (which is a relative effective divisor). Suppose moreover that  $x = \alpha(s) \in H_s$ . Let  $\zeta$  be the generic point of an irreducible component of  $H$  passing through  $x$ . Then  $\ell(\alpha^*\mathcal{M}) \geq \ell + 1$ . Moreover, if the equality  $\ell(\alpha^*\mathcal{M}) = \ell + 1$  holds, then (a)  $\mathcal{M}$  is Cohen-Macaulay at  $x$ ; (b)  $H$  is regular at  $x$ , and  $\mathcal{M}$  is of length 1 at  $\zeta$ ; (c)  $H$  cuts  $\alpha(S)$  transversally at  $x$ .*

Before proving this result consider the following technical Lemma.

**Lemma 3.2.2.** *Let  $Z = \text{Spec}(\mathbf{A})$  be a local noetherian regular scheme of dimension 2, and  $\mathcal{M}$  a torsion coherent  $\mathcal{O}_Z$ -module such that  $\text{Supp}(\mathcal{M})$  is of dimension 1. Let  $H_1, \dots, H_n$  be the reduced irreducible components of  $\text{Supp}(\mathcal{M})$ . Denote by  $\xi_i$  the generic point of  $H_i$ , and by  $\ell_i$  the length of  $\mathcal{M}_{\xi_i}$  over  $\mathcal{O}_{Z, \xi_i}$ . Let finally  $f \in \mathbf{A}$  be an element, which is part of a system of regular parameters of  $\mathbf{A}$ , such that  $Z_1 := V(f) \subset Z$  is not contained in  $\text{Supp}(\mathcal{M})$ . Then  $\ell(\mathcal{M}/f\mathcal{M}) \geq \sum_{i=1}^n \ell_i$ , with equality if and only if the following conditions are satisfied: (i) for each  $i$ , the scheme  $H_i$  is regular, and cuts  $Z_1$  transversally in  $Z$ ; (ii) the  $\mathcal{O}_Z$ -module  $\mathcal{M}$  is Cohen-Macaulay.*

*Proof.* Remark first that a coherent  $\mathcal{O}_Z$ -module  $\mathcal{N}$  with one dimensional support is Cohen-Macaulay if and only if  $\mathcal{N}$  has no embedded associated points. Indeed, suppose first that  $\mathcal{N}$  has no embedded associated points. Let  $\mathfrak{P}_1, \dots, \mathfrak{P}_r \subset \mathbf{A}$  be the minimal ideals of the support of  $\mathcal{N}$ , and let  $f \in \mathfrak{m}_{\mathbf{A}} \setminus \cup_i \mathfrak{P}_i$  (where  $\mathfrak{m}_{\mathbf{A}} \subset \mathbf{A}$  is the maximal ideal). Then multiplication by  $f$  provides an injective map (Corollary 1 of Proposition 7, Chapter I, § B [20])

$$\mathcal{N} \rightarrow \mathcal{N}, \quad n \mapsto f \cdot n$$

Hence, the maximal  $M$ -sequence of  $\mathcal{N}$  has at least  $1 = \dim(\mathcal{N})$  element, which implies that  $\mathcal{N}$  is Cohen-Macaulay (Definition 1 in § B.1 [20]). The converse statement follows from Proposition

13 of § B.2 in [20]. In order to prove the Lemma, we use induction on  $n$ . Let us begin with the case where  $n = 1$ . Denote by  $\xi = \xi_1$  the generic point of  $\text{Supp}(\mathcal{M})_{\text{red}} = H_1 = H$ , and by  $\ell = \ell_1$  the length of  $\mathcal{M}$  at  $\xi$ . Hence,  $\mathcal{M}_\xi$  has a filtration by  $\mathcal{O}_{Z,\xi}$ -submodules:

$$0 = \mathcal{M}_{\xi,0} \subset \mathcal{M}_{\xi,1} \subset \cdots \subset \mathcal{M}_{\xi,\ell} = \mathcal{M}_\xi,$$

where the successive quotients are isomorphic to  $k(\xi)$ . We then define  $\mathcal{M}_i$  as the inverse image of  $\mathcal{M}_{\xi,i}$  via the canonical map  $\mathcal{M} \rightarrow \mathcal{M}_\xi$ , thus obtaining a filtration on  $\mathcal{M}$ :

$$\mathcal{M}_{-1} := 0 \subseteq \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_\ell = \mathcal{M}.$$

In general,  $\mathcal{M}_0 \neq 0$ , and it is trivial if and only if  $\mathcal{M}$  has no embedded associated points. For each  $i \geq 0$ , let  $\mathcal{C}_i = \mathcal{M}_i / \mathcal{M}_{i-1}$ , which has no embedded associated point by definition whenever  $i \geq 1$ . Moreover,  $\mathcal{C}_{i,\xi} \simeq \mathcal{M}_{\xi,i} / \mathcal{M}_{\xi,i-1}$ , hence  $\mathcal{C}_{0,\xi} = 0$  and for  $i \geq 1$ , we have  $\mathcal{C}_{i,\xi} \simeq k(\xi)$ . In particular, if  $i \geq 1$ , we have  $\mathcal{C}_i \neq 0$  with schematic support equal to  $H = \text{Supp}(\mathcal{M})_{\text{red}}$ . Indeed, we only need to show that the schematic support  $\text{Supp}(\mathcal{C}_i) = V(\text{Ann}(\mathcal{C}_i))$  is *reduced*. Since  $\mathcal{C}_{i,\xi} \simeq k(\xi)$ ,  $\text{Supp}(\mathcal{C}_i)$  is reduced at  $\xi$ , hence it is generically reduced. Furthermore, since  $\mathcal{C}_i$  has no embedded associated points, so too is the scheme  $\text{Supp}(\mathcal{C}_i)$ . So  $\text{Supp}(\mathcal{C}_i)$  is reduced, and hence is equal to  $H$  as subscheme of  $Z$ . On the other hand, for  $i \geq 1$ , since the  $\mathcal{O}_Z$ -module  $\mathcal{C}_i$  has no embedded associated points, and  $Z_1 = V(f)$  is not contained in  $H$ , the map “multiplication by  $f$ ”:

$$\mathcal{C}_i \rightarrow \mathcal{C}_i, x \mapsto f \cdot x$$

is injective for  $i \geq 1$ . From this, we get a filtration of  $\mathcal{M}/f\mathcal{M}$ :

$$0 \subseteq \mathcal{M}_0/f\mathcal{M}_0 \subset \mathcal{M}_1/f\mathcal{M}_1 \subset \cdots \subset \mathcal{M}_\ell/f\mathcal{M}_\ell = \mathcal{M}/f\mathcal{M},$$

where for each  $i \geq 1$ , the quotient of  $\mathcal{M}_i/f\mathcal{M}_i$  by  $\mathcal{M}_{i-1}/f\mathcal{M}_{i-1}$  is isomorphic to  $\mathcal{C}_i/f\mathcal{C}_i$  which is non zero since  $\mathcal{C}_i \neq 0$ . As a result,  $\ell(\mathcal{M}/f\mathcal{M}) \geq \ell$ . Moreover,  $\ell(\mathcal{M}/f\mathcal{M}) = \ell$ , if and only if the following two conditions are realized: (a)  $\mathcal{M}_0/f\mathcal{M}_0 = 0$  which means  $\mathcal{M}_0 = 0$  by Nakayama’s lemma; (b) for each  $i$  ( $1 \leq i \leq \ell$ ), the  $\mathcal{O}_Z$ -module  $\mathcal{C}_i/f\mathcal{C}_i$  is of length 1 over  $\mathcal{O}_Z/f\mathcal{O}_Z$ .

Now, suppose that  $\ell(\mathcal{M}/f\mathcal{M}) = \ell$ , or equivalently that the conditions (a) and (b) above are verified. We will prove that  $\mathcal{M}$  is Cohen-Macaulay, and the schematic support  $H$  of  $\mathcal{C}_i$  is regular and cuts the subscheme  $V(f) \hookrightarrow Z$  transversally. In fact, condition (a) implies that the  $\mathcal{O}_Z$ -module  $\mathcal{M}$  has no embedded associated points, in particular, is Cohen-Macaulay. On the other hand, suppose that  $\text{Ann}(\mathcal{C}_i) = (g) \subset A$  (hence  $H$  is defined by the equation  $g = 0$  in  $Z$ ), and let  $c \in \mathcal{C}_i$  be such that  $c \notin f\mathcal{C}_i$ . Condition (b) together with Nakayama’s Lemma imply that the  $\mathcal{O}_Z$ -module  $\mathcal{C}_i$  is generated by  $c$ . The morphism  $\mathcal{O}_Z \rightarrow \mathcal{C}_i = \mathcal{O}_Z \cdot c$  defined by  $\lambda \mapsto \lambda c$  is then surjective, with kernel the ideal  $(g) = \text{Ann}(\mathcal{C}_i) = \text{Ann}(c)$ . Therefore,  $\mathcal{O}_Z/(g, f) \simeq \mathcal{C}_i/f\mathcal{C}_i$  is of length 1 over  $\mathcal{O}_Z/f\mathcal{O}_Z$ . Hence  $\text{Supp}(\mathcal{C}_i) = \text{Supp}(\mathcal{M})_{\text{red}} = H = V(g)$  is regular and cuts  $V(f) \hookrightarrow Z$  transversally. Conversely, suppose that  $\mathcal{M}$  is Cohen-Macaulay and that the scheme  $H$  is regular and cuts  $V(f) \hookrightarrow Z$  transversally. In particular,  $\mathcal{M}$  has no embedded associated point, which implies that  $\mathcal{M}_0 = 0$ , whence condition (a) holds. Moreover, since  $H = \text{Spec}(A/gA)$  is regular of dimension 1,  $A/gA$  is a principal ideal domain. Therefore, the  $\mathcal{O}_H = \mathcal{O}_Z/g\mathcal{O}_Z$ -module  $\mathcal{C}_i$  is free of rank 1. Hence,

$$\ell(\mathcal{C}_i/f\mathcal{C}_i) = \ell(A/(f, g)) = 1$$

since  $H = V(g) \hookrightarrow Z$  cuts  $V(f) \hookrightarrow Z$  transversally. In this way we get condition (b), which completes the proof of the Lemma in the case  $n = 1$ .



Suppose now that the assertion of the lemma has been verified for  $n - 1 \geq 1$ . Let  $\mathcal{M}' \subset \mathcal{M}$  be the sub-module defined as the kernel of the following map

$$\mathcal{M} \rightarrow \bigoplus_{i=2}^n \iota_{i,*} \iota_i^* \mathcal{M}$$

with  $\iota_i: \text{Spec}(k(\xi_i)) \rightarrow Z$  the canonical map and define  $\mathcal{M}''$  by the following exact sequence:

$$0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0.$$

Then  $\mathcal{M}''$  has no embedded associated points (and so is Cohen-Macaulay) and has support  $\bigcup_{i=2}^n H_i$ , while  $\mathcal{M}'$  has support  $H_1$ . One then has the following exact sequence (since  $\mathcal{M}''$  Cohen-Macaulay and  $V(f) \not\subseteq \text{Supp}(\mathcal{M}'')$ ):

$$0 \rightarrow \mathcal{M}'/f\mathcal{M}' \rightarrow \mathcal{M}/f\mathcal{M} \rightarrow \mathcal{M}''/f\mathcal{M}'' \rightarrow 0.$$

Hence, we have  $\ell(\mathcal{M}/f\mathcal{M}) = \ell(\mathcal{M}'/f\mathcal{M}') + \ell(\mathcal{M}''/f\mathcal{M}'')$ . Moreover, by the definitions of  $\mathcal{M}'$  and  $\mathcal{M}''$ , we have

$$\mathcal{M}'_{\xi_1} \simeq \mathcal{M}_{\xi_1}, \quad \text{and } \mathcal{M}''_{\xi_i} \simeq \mathcal{M}_{\xi_i} \text{ for } i = 2, \dots, n.$$

By applying the induction hypothesis, we get

$$\ell(\mathcal{M}/f\mathcal{M}) = \ell(\mathcal{M}'/f\mathcal{M}') + \ell(\mathcal{M}''/f\mathcal{M}'') \geq \ell(\mathcal{M}'_{\xi_1}) + \sum_{i=2}^n \ell(\mathcal{M}''_{\xi_i}) = \sum_{i=1}^n \ell_i,$$

with equality if and only if  $\ell(\mathcal{M}'/f\mathcal{M}') = \ell_1$ , and  $\ell(\mathcal{M}''/f\mathcal{M}'') = \sum_{i=2}^n \ell_i$ . In other words, equality holds if and only if (a)  $\mathcal{M}', \mathcal{M}''$  are Cohen-Macaulay, and (b) the subschemes  $H_i$  are regular cutting  $Z_1$  transversally in  $Z$ . Since  $\mathcal{M}''$  is already Cohen-Macaulay, condition (a) is equivalent to saying that  $\mathcal{M}$  is Cohen-Macaulay. This completes the proof of the Lemma.  $\square$

*Proof of 3.2.1.* Since  $\alpha: S \rightarrow Z$  is a section of  $Z/S$ , there exist elements  $f_1, \dots, f_d$  of  $\mathcal{O}_{Z,x}$  which generate, together with  $\pi$ , the maximal ideal of  $\mathcal{O}_{Z,x}$  and  $\alpha(S) = V(f_1, \dots, f_d) \hookrightarrow Z$ . Up to replacing  $Z$  by its localization at  $x$ , we may assume that  $Z$  is the spectrum of a regular local ring of dimension  $d + 1$ . In particular  $Z_s = \overline{\{\xi\}}$  is regular and irreducible. We will prove the Proposition by induction on  $d$ . The case  $d = 0$ , i.e.,  $S = Z$ , is trivial. We start illustrating the case  $d = 1$ . In this case,  $Z$  is a 2-dimensional local regular scheme, with  $\mathcal{M}$  a torsion coherent module on  $Z$ . When  $\ell = 0$ , the conclusion of (1) is clear since in this situation, we always have  $\ell(\alpha^* \mathcal{M}) \geq \ell = 0$ , and an equality means that  $x \notin \text{Supp}(\mathcal{M})$ , or equivalently,  $\mathcal{M}_x = 0$ . In fact, here we have  $\mathcal{M} = 0$  since  $x \in Z$  is the only closed point of the local scheme  $Z$ . To finish the proof of (1) for  $d = 1$ , we may assume that  $\ell \geq 1$ . Since  $\xi \in \text{Supp}(\mathcal{M})$ , the closed subscheme  $Z_s = \overline{\{\xi\}} \subseteq \text{Supp}(\mathcal{M})$  is one of the irreducible components of  $\text{Supp}(\mathcal{M})$  in  $Z$ . Now by applying Lemma 3.2.2, we get  $\ell(\alpha^* \mathcal{M}) \geq \ell$ . The equality holds if and only if  $Z_s$  is the only component of  $Z$ ,  $Z_s$  cuts  $\alpha(S)$  transversally in  $Z$  and  $\mathcal{M}$  is Cohen-Macaulay at  $x$ . We now consider assertion (2). By assumption,  $\text{Supp}(\mathcal{M})$  is the union of the one dimensional subscheme  $H$  with, possibly,  $Z_s$  if  $\ell > 0$ ; hence  $\ell(\alpha^* \mathcal{M}) \geq \ell + 1$ . If  $\ell(\alpha^* \mathcal{M}) = \ell + 1$ , on applying Lemma 3.2.2 once again, we see that  $\mathcal{M}$  is Cohen-Macaulay at  $x$ , the subscheme  $H \subset Z$  is irreducible, regular and cuts  $\alpha(S)$  transversally at  $x$ . Moreover,  $\mathcal{M}$  must have length 1 at the generic point of  $H$ . This proves (2), and hence the Proposition, for  $d = 1$ .

For the general case, denote by  $Z_1 \hookrightarrow Z$  the closed subscheme defined by the ideal  $(f_1)$ , by  $\xi_1$  the generic point of  $Z_{1,s}$  and by  $\mathcal{M}_1$  the pull-back of  $\mathcal{M}$  to  $Z_1$ . Then  $Z_1$  is a regular local

scheme of dimension  $d$ , which is not contained in the support of  $\mathcal{M}$ . In particular,  $\mathcal{M}_1$  is again a torsion coherent sheaf on  $Z_1$ . The morphism  $\alpha: S \rightarrow Z$  factors through  $Z_1 \hookrightarrow Z$ , and we denote by  $\alpha_1: S \rightarrow Z_1$  the morphism obtained in this way. In particular,  $\alpha^*\mathcal{M} \simeq \alpha_1^*\mathcal{M}_1$  and  $\alpha(S) \not\subseteq \text{Supp}(\mathcal{M}_1)$ . Hence, in order to prove the first assertion of (1), we only need to verify that the module  $\mathcal{M}_1$  is of length  $\geq \ell$  at  $\xi_1$  and then apply the induction hypothesis. To see the inequality  $\ell(\mathcal{M}_{1,\xi_1}) \geq \ell$ , since

$$\mathcal{M}_{1,\xi_1} \simeq \mathcal{M}_{\xi_1}/f_1\mathcal{M}_{\xi_1},$$

we are reduced to showing that the restriction of the torsion sheaf  $\widetilde{\mathcal{M}} := \mathcal{M}|_{\text{Spec}(\mathcal{O}_{Z,\xi_1})}$  to the subscheme  $\text{Spec}(\mathcal{O}_{Z,\xi_1}/f_1\mathcal{O}_{Z,\xi_1}) \hookrightarrow \text{Spec}(\mathcal{O}_{Z,\xi_1}) =: \widetilde{Z}$  is of length  $\geq \ell$ . By definition,  $\xi \in Z_s$  is contained in the special fibre  $\widetilde{Z}_s$  of  $\widetilde{Z}$ , and  $\widetilde{\mathcal{M}}$  has length  $\ell$  at  $\xi \in \widetilde{Z}_s$ . Hence, we need only apply Lemma 3.2.2 to the two dimensional regular local scheme  $\widetilde{Z}$  to get the conclusion. We can also summarize the previous arguments by the following relations:

$$(26) \quad \ell(\alpha^*\mathcal{M}) = \ell(\alpha_1^*\mathcal{M}_1) \geq \ell(\mathcal{M}_{1,\xi_1}) = \ell(\widetilde{\mathcal{M}}/f_1\widetilde{\mathcal{M}}) \geq \ell(\widetilde{\mathcal{M}}_\xi) = \ell(\mathcal{M}_\xi) = \ell.$$

Next, we examine the condition  $\ell(\alpha^*\mathcal{M}) = \ell$ . By (26), we have  $\ell(\alpha^*\mathcal{M}) = \ell$  if and only if (a)  $\ell(\widetilde{\mathcal{M}}/f_1\widetilde{\mathcal{M}}) = \ell(\widetilde{\mathcal{M}}_\xi) = \ell$ , and (b)  $\ell(\mathcal{M}_{1,\xi_1}) = \ell(\alpha_1^*\mathcal{M}_1)$ . Consider the conditions (a'):  $\mathcal{M}$  is Cohen-Macaulay having support contained in  $Z_s$  at  $\xi_1$  and (b'):  $\mathcal{M}_1$  is Cohen-Macaulay, with support contained in  $Z_{1,s}$  at  $x$ . On applying the induction hypothesis to the torsion module  $\mathcal{M}_1$  on the  $d$ -dimensional scheme  $Z_1$ , one checks immediately that condition (b) is equivalent to condition (b'). Furthermore on applying the induction hypothesis to the torsion module  $\widetilde{\mathcal{M}} = \mathcal{M}|_{\text{Spec}(\mathcal{O}_{Z,\xi_1})}$  on the 2-dimensional local scheme  $\widetilde{Z} = \text{Spec}(\mathcal{O}_{Z,\xi_1})$ , and since  $\widetilde{Z}$  is the localization of  $Z$  at the point  $\xi_1 \in Z$ , we find that conditions (a) and (a') are equivalent. Hence  $\ell(\alpha^*\mathcal{M}) = \ell$  if and only if (a') and (b') hold.

Now, we proceed with the proof of the second part of (1). Suppose first  $\ell(\alpha^*\mathcal{M}) = \ell$ , or equivalently, that the previous conditions (a') and (b') hold, and prove that  $\mathcal{M}$  is Cohen-Macaulay with support contained in  $Z_s$  at  $x$ . We first claim that the multiplication by  $f_1$  on  $\mathcal{M}$  provides an injective map. To see this fact, let  $\mathcal{M}'$  be the submodule of  $\mathcal{M}$  formed by the elements killed by a power of  $f_1$ , and let  $\mathcal{M}'' = \mathcal{M}/\mathcal{M}'$ . By definition,  $\text{Supp}(\mathcal{M}') \subset Z_1 \cap \text{Supp}(\mathcal{M})$ , which is hence of codimension at least 2. By definition of  $\mathcal{M}''$ , the multiplication by  $f_1$  on  $\mathcal{M}''$  is an injective map, hence the canonical map

$$(27) \quad \mathcal{M}'/f_1\mathcal{M}' \rightarrow \mathcal{M}/f_1\mathcal{M} = \mathcal{M}_1$$

is injective. On one hand, condition (a') above implies that  $\xi_1 \notin \text{Supp}(\mathcal{M}')$  (since  $\text{Supp}(\mathcal{M}') \subset V(f_1) \cap Z_s = Z_{1,s} \subset Z_s$  and  $\mathcal{M}$  is Cohen-Macaulay at  $\xi_1$  by (a')), hence  $\xi_1 \notin \text{Supp}(\mathcal{M}'/f_1\mathcal{M}')$ . In particular,  $\text{Supp}(\mathcal{M}'/f_1\mathcal{M}') \cap Z_{1,s} \subsetneq Z_{1,s}$ . On the other hand, condition (b') and the injection (27) imply that the support of  $\mathcal{M}'/f_1\mathcal{M}'$  is contained in  $Z_{1,s}$ . Hence  $\mathcal{M}'/f_1\mathcal{M}' = 0$ , and then  $\mathcal{M}' = 0$  by Nakayama's Lemma. As a result, the multiplication by  $f_1$  on  $\mathcal{M}$  is injective. Moreover the quotient sheaf  $\mathcal{M}_1 = \mathcal{M}/f_1\mathcal{M}$  is Cohen-Macaulay of dimension  $d-1 = \dim(Z_1) - 1$  by (b'), hence also  $\mathcal{M}$  is Cohen-Macaulay. To see that the support of  $\mathcal{M}$  is contained in  $Z_s$ , suppose  $\text{Supp}(\mathcal{M})$  contains a component  $\Gamma$  different from  $Z_s$  at the point  $x$ . Since we have shown that  $\mathcal{M}$  is Cohen-Macaulay at  $x$ , it follows that  $\Gamma$  is also of codimension 1. By condition (b) above,  $\mathcal{M}_1$  has support contained in  $Z_{1,s}$  at  $x$ . So  $\Gamma \cap Z_1 \subset Z_{1,s}$  which is in fact an equality of sets for reasons of dimension. As a result, one finds that  $\xi_1 \in \Gamma$ , which means that  $\text{Supp}(\mathcal{M})$  has at least two components (of codimension 1) at  $\xi_1$ , but this is impossible because of the condition (a'). This proves that  $\mathcal{M}$  is Cohen-Macaulay with support contained in  $Z_s$  at  $x$ . Conversely, suppose  $\mathcal{M}$  is Cohen-Macaulay with support contained in  $Z_s$  at  $x$ . We must prove that  $\ell(\alpha^*\mathcal{M}) = \ell$ . First

of all, this condition implies in particular that  $\mathcal{M}$  is Cohen-Macaulay with support contained in  $Z_s$  at  $\xi_1$ , namely condition (a') holds. To complete the proof of (1), we only need to show that condition (b') also holds. It is clear that the support of  $\mathcal{M}_1$  is contained in  $Z_{1,s}$ , so we need only verify that  $\mathcal{M}_1$  is Cohen-Macaulay. Since  $Z_{1,s} = \text{Supp}(\mathcal{M}_1)$  has dimension equal to  $\dim(\mathcal{M}) - 1$ ,  $\mathcal{M}_1$  is also Cohen-Macaulay ([20] Chapter IV § B.2, Proposition 14). This finishes the proof of (1).

To finish the proof of (2), since this is a local question for the étale topology on  $S$ , we may assume that  $S$  is strictly local, in particular the residue field is an infinite set. This implies that the residue field  $k(x)$  of  $Z$  at  $x$  is also infinite. Since  $k(x)$  is an infinite field, up to replacing  $f_1$  by  $f_1 + \lambda f_2$ , for a suitable  $\lambda \in \mathcal{O}_{Z,x}^*$ , we may assume that  $Z_{1,s} \not\subseteq H_s$ , so that  $H_{1,s} = H_s \cap Z_{1,s} \hookrightarrow Z_{1,s}$  is of codimension 1 in  $Z_{1,s}$  (where  $H_1 := H \cap Z_1$ ).

As we have seen in (26),  $\mathcal{M}_1$  has length  $\geq \ell$  at  $\xi_1$ . Since  $x \in H_{1,s}$ , on applying the induction hypothesis to  $Z_1$ , we find that

$$(28) \quad \ell(\alpha^* \mathcal{M}) = \ell(\alpha_1^* \mathcal{M}_1) \geq \ell(\mathcal{M}_{1,\xi_1}) + 1 \geq \ell + 1.$$

This proves the first assertion in (2). From now on, we suppose that  $\ell(\alpha^* \mathcal{M}) = \ell + 1$ . According to (28), we get  $\ell(\alpha^* \mathcal{M}) = \ell(\alpha_1^* \mathcal{M}_1) = \ell + 1$ , and  $\mathcal{M}_1$  is of length  $\ell$  at  $\xi_1$ . By the induction hypothesis, we have (i)  $H_{1,\text{red}}$  is irreducible and regular, and moreover  $\alpha_1(S)$  cuts  $H_1$  transversally in  $Z_1$  at  $x$ ; (ii)  $\mathcal{M}_1$  is Cohen-Macaulay at  $x$  in  $Z_1$ , and if we denote by  $\zeta_1 \in H_1$  the generic point of  $H_1$ , then  $\mathcal{M}_1$  is of length 1 at  $\zeta_1$ . Denote by  $Z'$  the localization of  $Z$  at  $\zeta_1$ , and by  $\mathcal{M}'$  the inverse image of  $\mathcal{M}$  by the canonical morphism  $Z' \rightarrow Z$ . Then,  $\mathcal{M}'/f_1 \mathcal{M}'$  is of length 1 over  $\mathcal{O}_{Z'}/f_1 \mathcal{O}_{Z'}$ . Hence, on applying Lemma 3.2.2 to the torsion module  $\mathcal{M}|_{\text{Spec}(\mathcal{O}_{Z,\zeta_1})}$  on the two dimension regular local scheme  $\text{Spec}(\mathcal{O}_{Z,\zeta_1})$  we get that  $H$  is regular at  $\zeta_1$ , and it cuts  $Z_1$  transversally at  $\zeta_1$ . Moreover,  $\mathcal{M}$  is Cohen-Macaulay with support contained in  $H$  at  $\zeta_1$ , and  $\mathcal{M}$  is of length 1 at the generic point  $\zeta$  of  $H$ . Using now the fact that  $H_1 = H \cap Z_1$  is irreducible, we find that  $H$  itself must be irreducible, since otherwise,  $H$  would have at least two components at  $\zeta_1$ . Therefore,  $H_1$  is generically reduced. But  $H_1$  is a divisor inside a regular scheme  $Z_1$ , hence  $H_1$  is Cohen-Macaulay and thus  $H_1$  is reduced. By assertion (i), we find that  $H$  is irreducible and regular, cutting  $Z_1$  transversally at  $x$  inside  $Z$ . Now we need only verify that  $\mathcal{M}$  is Cohen-Macaulay on  $Z$ . By (ii), we need only show that  $\mathcal{M}$  has no embedded associated points. Denote by  $\mathcal{N}'$  the biggest quotient without embedded associated points of  $\mathcal{M}$ , and denote by  $\mathcal{N}$  the  $\mathcal{O}_Z$ -submodule defined by the following exact sequence

$$0 \rightarrow \mathcal{N} \rightarrow \mathcal{M} \rightarrow \mathcal{N}' \rightarrow 0;$$

we have the short exact sequence

$$0 \rightarrow \mathcal{N}/f_1 \mathcal{N} \rightarrow \mathcal{M}/f_1 \mathcal{M} \rightarrow \mathcal{N}'/f_1 \mathcal{N}' \rightarrow 0.$$

According to Nakayama's Lemma, to complete the proof, we need only show that  $\mathcal{N}/f_1 \mathcal{N} = 0$ . By definition,  $\text{Supp}(\mathcal{N}') = \text{Supp}(\mathcal{M})$ , and  $\ell(\mathcal{N}'_{\xi}) = \ell(\mathcal{M}_{\xi}) = \ell$ . Hence, according to the first part of (2) (which has already been proved), we have  $\ell(\alpha^* \mathcal{N}') \geq \ell + 1$ . On the other hand, there is a surjection  $\alpha^* \mathcal{M} \rightarrow \alpha^* \mathcal{N}'$ , whence we have  $\ell + 1 = \ell(\alpha^* \mathcal{M}) \geq \ell(\alpha^* \mathcal{N}')$ . As a result, we have  $\ell(\alpha^* \mathcal{N}') = \ell + 1$ . Hence the  $\mathcal{O}_Z$ -module  $\mathcal{N}'$  again satisfies the assumptions of the assertion (2) of this Proposition: On applying what was proved a few lines above to the torsion sheaf  $\mathcal{N}'$  in place of  $\mathcal{M}$ , we get  $\ell((\mathcal{N}'/f_1 \mathcal{N}')_{\zeta_1}) = 1$ , and  $\ell((\mathcal{N}'/f_1 \mathcal{N}')_{\xi_1}) = \ell$ . Hence  $(\mathcal{N}/f_1 \mathcal{N})_{\zeta_1} = (\mathcal{N}/f_1 \mathcal{N})_{\xi_1} = 0$ . As a result, the support of  $\mathcal{N}/f_1 \mathcal{N}$  is of dimension  $< d - 2$ . But we have seen that  $\mathcal{M}_1 = \mathcal{M}/f_1 \mathcal{M}$  is Cohen-Macaulay with support  $H_1 \cup Z_{1,s}$  of dimension  $d - 2$ , hence we must have  $\mathcal{N}/f_1 \mathcal{N} = 0$ . This completes the proof.  $\square$

### 3.3 Preliminaries on the comparison between the pro-algebraic structures

The aim of this section is to show that the canonical map of sheaves  $q: \text{Pic}_{X/S}^0 \rightarrow J$  in (10) induces, for each  $n \geq 1$ , a morphism of smooth algebraic  $k$ -groups

$$(29) \quad q_n: \text{Gr}(\text{P}_{[\psi(n)]}) = \text{Gr}(\text{Pic}_{X_{\psi(n)/S}}^0) \rightarrow \text{Gr}_n(J),$$

and that the maps  $q_n$  are compatible in the evident way.

Let  $n \geq 1$  be an integer. We have seen in § 3.1 that there exists a morphism of sheaves  $\text{Pic}_{X_{nd}/S_n}^0 \rightarrow J \times_S S_n$ . In this way we get a morphism of smooth algebraic  $k$ -groups

$$(30) \quad q'_n: \text{Gr}(\text{P}_{[nd]}) \rightarrow \text{Gr}_n(J).$$

Since  $\psi(n) \leq nd$  (cf. § 2.2), there is a canonical morphism of smooth algebraic  $k$ -groups

$$\text{Gr}(\text{P}_{[nd]}) \rightarrow \text{Gr}(\text{P}_{[\psi(n)]}).$$

So, in order to prove the existence of  $q_n$  as above, it suffices to verify that the morphism  $q'_n$  in (30) factors as follows:

$$\begin{array}{ccc} \text{Gr}(\text{P}_{[nd]}) & \xrightarrow{q'_n} & \text{Gr}_n(J) \\ \downarrow & \searrow q_n & \\ \text{Gr}(\text{P}_{[\psi(n)]}) & & \end{array}$$

On the other hand, since the morphism of algebraic  $k$ -groups  $\text{Gr}(\text{P}_{[nd]}) \rightarrow \text{Gr}(\text{P}_{[\psi(n)]})$  has smooth kernel (see Lemma 3.1.1) and  $k$  is algebraically closed, *we need only check the factorization on the level of  $k$ -points*. Since the maps  $\text{Pic}^0(X) \rightarrow \text{Pic}^0(X_{\psi(n)}) = \text{Gr}(\text{P}_{[\psi(n)]})(k)$  are surjective, the verification is reduced to proving the existence of the following factorization:

$$\begin{array}{ccc} \text{Pic}^0(X) & \twoheadrightarrow & \text{Pic}^0(X_{\psi(n)}) \\ q \downarrow & & \downarrow q_n \\ J(S) & \twoheadrightarrow & J(S_n) \end{array}$$

where we again denote by  $q_n$  the map induced by (29) on  $k$ -points. With the help of the rigidified Picard functor we will establish this factorization via induction on  $n$ .

### 3.4 Comparison of the pro-algebraic structures

In the following discussion we fix a rigidificator  $Y \hookrightarrow X$  of the relative Picard functor  $\text{Pic}_{X/S}$ , and for simplicity, we denote by  $G = (\text{Pic}_{X/S}, Y)^0$  the identity component of the rigidified Picard functor of  $X/S$  along  $Y/S$ . As usual  $J$  denotes the identity component of the Néron model of  $\text{Pic}_{X_K/K}^0$  over  $S$ . According to Proposition 3.2 in [11],  $G$  is representable by a smooth separated  $S$ -group scheme. Consider the canonical morphism of  $S$ -group schemes  $r: G = (\text{Pic}_{X/S}, Y)^0 \rightarrow \text{Pic}_{X/S}^0$  (recalled in § 1.1), which is surjective for the étale topology. Let  $H$  be the schematic closure of  $\ker(r_K) \subset G_K$  in  $G$ . It is a flat  $S$ -group scheme of finite type with smooth generic fibre, which is also the kernel of the canonical morphism  $\theta: G \rightarrow J$  (composition of  $r: G \rightarrow \text{Pic}_{X/S}^0$  and the epimorphism  $q: \text{Pic}_{X/S}^0 \rightarrow J$ ; see for example [16], 4.1, for the fact that  $\ker(\theta) = H$ ). Since  $S$  is strictly local, the morphism  $r$  induces a surjective map (still denoted by  $r$ ) between the  $S$ -sections:

$$r: G(S) \rightarrow \text{Pic}^0(X).$$

Let  $\mathfrak{L}$  be the (rigidified) Poincaré sheaf on  $X \times G$ . For each  $n \in \mathbf{Z}_{\geq 1}$ , we denote by

$$r_n: G(S) \xrightarrow{r} \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0(X_{\psi(n)})$$

the composition of maps which sends  $\varepsilon \in G(S)$  to  $\mathcal{L}_\varepsilon|_{X_{\psi(n)}} \in \mathrm{Pic}^0(X_{\psi(n)})$ , where  $\mathcal{L}_\varepsilon$  is the sheaf  $(\mathrm{id}_X \times \varepsilon)^* \mathfrak{L}$ . These maps are all surjective since  $\mathcal{O}_K$  is strictly henselian.

Let  $p_G: X \times_S G \rightarrow G$  be the projection onto the second factor, and consider the object  $\mathrm{R}p_{G,*} \mathfrak{L}$  in the derived category of  $\mathcal{O}_S$ -modules. It is well known that this complex is quasi-isomorphic to a perfect complex of perfect amplitude contained in  $[0, 1]$ , i.e., locally for the Zariski topology on  $G$ ,  $\mathrm{R}p_{G,*} \mathfrak{L}$  can be represented by a complex

$$\dots \longrightarrow \mathcal{F}^0 \xrightarrow{u} \mathcal{F}^1 \longrightarrow \dots$$

with  $\mathcal{F}^i$  ( $i = 0, 1$ ) locally free  $\mathcal{O}_G$ -modules of the same rank (EGA III 6.10.5). The cokernel  $\mathcal{M}$  of  $u$  gives the  $\mathcal{O}_G$ -module  $\mathrm{R}^1 p_{G,*} \mathfrak{L}$ , and for any section  $\varepsilon: S \rightarrow G$  of  $G/S$ , the pull-back  $\varepsilon^* \mathcal{M}$  is given by the cohomology group  $H^1(X, \mathcal{L}_\varepsilon)$ . On the other hand, for  $L$  an invertible sheaf of degree 0 on  $X_K$ ,  $H^1(X_K, L) \neq 0$  if and only if  $L \simeq \mathcal{O}_{X_K}$ . Therefore, the morphism  $u$  above is injective and  $\det(u) \neq 0$ . Hence  $\mathcal{M}$  is a torsion  $\mathcal{O}_G$ -module which admits a resolution of length 1 by locally free  $\mathcal{O}_G$ -modules. In particular,  $\mathcal{M}$  is Cohen-Macaulay, with support  $\mathrm{Supp}(\mathcal{M}) \subset G$  purely of codimension 1 satisfying the inclusion relations (of sets)  $H \subset \mathrm{Supp}(\mathcal{M}) \subset H \cup G_s$ .

**Lemma 3.4.1.** *Let notations be as above. Then  $\mathrm{Supp}(\mathcal{M}) = H$  as sets.*

*Proof.* Let  $\xi$  be the generic point of  $G_s$ , and  $\ell$  the length of the  $\mathcal{O}_{G,\xi}$ -module  $\mathcal{M}_\xi$ . We will first prove by contradiction that  $\ell = 0$ . Suppose then  $\ell \geq 1$ . Let  $\varepsilon \in G(S)$  be a section of  $G$ , and let  $\mathcal{L}_\varepsilon = (\mathrm{id}_X \times \varepsilon)^* \mathfrak{L}$ . According to Proposition 3.2.1, the  $\mathcal{O}_K$ -module  $H^1(X, \mathcal{L}_\varepsilon) \simeq \varepsilon^* \mathcal{M}$  is of length at least  $\ell \geq 1$ . By Corollary 2.2.3, this is equivalent to saying that  $\mathcal{L}_\varepsilon|_{X_1} \simeq \mathcal{I}^i|_{X_1}$  with  $i$  a suitable integer. This last fact implies that the surjective homomorphism

$$r_1: G(S) \rightarrow \mathrm{Pic}^0(X_1), \varepsilon \mapsto \mathcal{L}_\varepsilon|_{X_1}$$

has finite image. However, this produces a contradiction since  $k$  is algebraically closed and so  $\mathrm{Pic}^0(X_1) \simeq \mathrm{Pic}_{X_1/k}^0(k)$  is an infinite group. Therefore  $\mathcal{M}_\xi = 0$ . In particular,  $\xi \notin \mathrm{Supp}(\mathcal{M})$ , which completes the proof of the Lemma since  $\mathrm{Supp}(\mathcal{M}) \subset G$  is purely of codimension 1.  $\square$

Let us begin the comparison of the two filtrations defined in § 3.1 at the level  $n = 1$ . Since  $X_1/S$  can be defined over the closed point  $s$  of  $S$ , its Picard functor  $\mathrm{P}_{[1]} = \mathrm{Pic}_{X_1/S}^0$  can also be defined over  $s$ . Hence, by adjunction, the morphism of functors  $r_1: G \rightarrow \mathrm{P}_{[1]}$  corresponds to a morphism of algebraic groups over the closed points  $s$  of  $S$ :

$$r_{1,s}: G_s \rightarrow \mathrm{P}_{[1],s} = \mathrm{Pic}_{X_1/k}^0,$$

which renders the following diagram commutative

$$\begin{array}{ccccc} G & \longrightarrow & \mathrm{Pic}_{X/S}^0 & \longrightarrow & \mathrm{P}_{[1]} = i_* \mathrm{P}_{[1],s} \\ \downarrow & & & \nearrow i_* r_{1,s} & \\ i_* G_s & & & & \end{array}$$

Let  $x \in G_s(k)$  be a closed point,  $\varepsilon \in G(S)$  a section lifting  $x$  and put  $\mathcal{L}_\varepsilon = (\mathrm{id}_X \times \varepsilon)^* \mathfrak{L}$ . This is a rigidified invertible sheaf over  $X$ . Since  $\mathrm{Supp}(\mathcal{M}) = H$  (Lemma 3.4.1), one finds that

$x = \varepsilon(s) \in H_s(k)$  if and only if the  $\mathcal{O}_K$ -module  $\varepsilon^* \mathcal{M} = H^1(X, \mathcal{L}_\varepsilon)$  is of length  $\geq 1$ . Moreover, in view of Lemma 2.2.3 this last condition is equivalent to saying that  $\mathcal{L}_\varepsilon|_{X_1} \simeq \mathcal{I}^i|_{X_1}$  for a suitable integer  $i$ . In particular, the image  $r_{1,s}(H_s(k))$  is a finite set of  $P_{[1],s}(k)$ , and the kernel of  $r_{1,s}$  is contained in  $H_s$ . Let  $Z$  be the schematic closure of the set of those points  $x \in G_s(k)$  which admit a lifting  $\varepsilon \in G(S)$  such that  $\mathcal{L}_\varepsilon|_{X_1} \simeq \mathcal{O}_{X_1}$ . By continuity of  $r_{1,s}$ , the subgroup scheme  $Z \subset G_s$  is a union of irreducible components of  $H_{s,\text{red}}$ . Denote by  $G^{[1]} \rightarrow G$  the dilatation of  $G$  with center  $Z \subset G_s$ . By definition of  $Z$ , we have an exact sequence of smooth  $k$ -group schemes

$$(31) \quad 0 \rightarrow Z \rightarrow G_s \xrightarrow{r_1} \text{Pic}_{X_1/k}^0 \rightarrow 0.$$

Moreover, according to the universal property of dilatations ([5], 3.2/1), we have the following short exact sequence of abstract groups:

$$(32) \quad 0 \rightarrow G^{[1]}(S) \rightarrow G(S) \rightarrow \text{Pic}^0(X_1) \rightarrow 0.$$

Consider now the morphism  $\theta: G \rightarrow J$ . Denote by  $G^{[1]}'$  the dilatation of  $G$  with center  $H_{s,\text{red}} = \ker(\theta)_{s,\text{red}} \hookrightarrow G_s$ . Since  $H_s$  is the kernel of the canonical map  $\theta_s: G_s \rightarrow J_s$ , the universal property of dilatations implies that the following sequence is exact

$$(33) \quad 0 \rightarrow G^{[1]}'(S) \rightarrow G(S) \rightarrow J(S_1) \rightarrow 0.$$

Since  $Z \subset H_{s,\text{red}}$  is an open subgroup,  $G^{[1]}$  is an open subgroup of  $G^{[1]}'$ . From the exact sequences (32), (33), we obtain a morphism of groups  $q_1: \text{Pic}^0(X_1) \rightarrow J(S_1)$  which makes the external square commute:

$$\begin{array}{ccccc} G(S) & \xrightarrow{r} & \text{Pic}^0(X) & \longrightarrow & \text{Pic}^0(X_1) \\ \parallel & & \downarrow q & & \downarrow q_1 \\ G(S) & \xrightarrow{\theta} & J(S) & \longrightarrow & J(S_1) \end{array}$$

Moreover, from the fact that  $q \circ r = \theta$  and the surjectivity of  $r$ , the square on the right also commutes. The morphism of abstract groups  $q_1$  is surjective (since all the other maps are), and has kernel generated by  $\mathcal{I}|_{X_1} \in \text{Pic}(X_1)$  (see Corollary 2.2.3). Note that the diagram above fits also into a bigger commutative diagram

$$(34) \quad \begin{array}{ccccccc} G^{[1]}(S) & \hookrightarrow & G[S] & \longrightarrow & \text{Pic}^0(X_1) & & \\ & \searrow & \downarrow \theta & \searrow r & \downarrow q_1 & & \\ & & P^{[1]}(S) & \hookrightarrow & \text{Pic}^0(X) & \longrightarrow & \text{Pic}^0(X_1) \\ & & \vdots \downarrow q^{[1]} & & \downarrow q & & \downarrow q_1 \\ & & J^{[1]}(S) & \hookrightarrow & J(S) & \longrightarrow & J(S_1) \end{array}$$

On the level of pro-algebraic groups we have then shown that the morphism  $q$  in (24) induces a map

$$(35) \quad q_1: \text{Gr}(\text{Pic}_{X_{\psi(1)}/S}^0) \rightarrow \text{Gr}_1(J)$$

because, as we noted in §3.3, it is sufficient to check the factorization on  $k$ -points.

In order to proceed with the comparison of the filtrations for higher  $n$ , let us denote by  $\mathcal{M}^{[1]}$  (respectively by  $\mathcal{M}^{[1]}'$ ) the inverse image of  $\mathcal{M}$  over  $G^{[1]}$  (respectively over  $G^{[1]}'$ ) via



the morphism  $G^{[1]} \rightarrow G$  (respectively via the morphism  $G^{[1]'} \rightarrow G$ ). Let  $H^{[1]}$  (respectively  $H^{[1]'}$ ) be the schematic closure of  $H_K \hookrightarrow G_K^{[1]} = G_K$  in  $G^{[1]}$  (respectively in  $G^{[1]'}$ ). Then  $\mathcal{M}^{[1]}$  (respectively  $\mathcal{M}^{[1]'}$ ) is a coherent torsion sheaf with support in  $H^{[1]} \cup G_s^{[1]}$  (respectively in  $H^{[1]'} \cup G_s^{[1]'}), which admits a resolution of length 1 by locally free  $\mathcal{O}_{G^{[1]}}$ -modules (respectively  $\mathcal{O}_{G^{[1]'}}$ -modules). In particular, since the schemes  $G^{[1]}$  and  $G^{[1]'}$  are regular,  $\mathcal{M}^{[1]}$  and  $\mathcal{M}^{[1]'}$  are Cohen-Macaulay as modules. On the other hand, by the universal property of dilatations, the composed morphism  $G^{[1]} \rightarrow G \rightarrow J$  (respectively  $G^{[1]'} \rightarrow G \rightarrow J$ ) factors through  $J^{[1]} \rightarrow J$ . We denote by  $\theta^{[1]}: G^{[1]} \rightarrow J^{[1]}$  (respectively by  $\theta^{[1]'}: G^{[1]'} \rightarrow J^{[1]}$ ) the morphism obtained in this way.$

**Lemma 3.4.2.** *Let the notation be as above.*

- (i) *Let  $\xi'_1$  be a generic point of  $G_s^{[1]'}$ , then the  $\mathcal{O}_{G^{[1]'}, \xi'_1}$ -module  $\mathcal{M}_{\xi'_1}^{[1]'}$  is of length 1.*
- (ii) *The scheme  $H$  is normal.*
- (iii) *The morphism  $\theta^{[1]}: G^{[1]} \rightarrow J^{[1]}$  induces a surjection  $G^{[1]}(S) \rightarrow J^{[1]}(S)$ . In particular,  $\theta^{[1]}$  is a faithfully flat morphism of  $S$ -group schemes, with  $\ker(\theta^{[1]}) = H^{[1]}$ .*

*Proof.* Observe first that we have a commutative diagram with exact rows, where the first row is (32):

$$\begin{array}{ccccccc} 0 & \longrightarrow & G^{[1]}(S) & \longrightarrow & G(S) & \longrightarrow & \mathrm{Pic}^0(X_1) \longrightarrow 0 \\ & & \downarrow & & \downarrow r_2 & & \parallel \\ 0 & \longrightarrow & P^{[1, \psi(2)]}(S) & \longrightarrow & P_{[\psi(2)]}(S) & \longrightarrow & P_{[1]}(S) \longrightarrow 0 \end{array}$$

The morphism  $G^{[1]}(S) \rightarrow P^{[1, \psi(2)]}(S)$  is surjective, since the map  $r_2$  is surjective. Moreover, by Corollary 2.1.9, the group  $P^{[1, \psi(2)]}(S)$  is an  $\mathcal{O}_K$ -module of length 1. Hence, it is an infinite group. Therefore, the composed morphism

$$G^{[1]}(S) \rightarrow G(S) \rightarrow P_{[\psi(2)]}(S) = \mathrm{Pic}^0(X_{\psi(2)}),$$

has infinite image. Hence, the composed morphism

$$(36) \quad G^{[1]'}(S) \rightarrow G(S) \rightarrow P_{[\psi(2)]}(S) = \mathrm{Pic}^0(X_{\psi(2)})$$

also has infinite image because  $G^{[1]}$  is an open subgroup of  $G^{[1]'}$ .

Next, we consider the map of functors  $r'_2: G^{[1]'} \rightarrow P_{[\psi(2)]} = \mathrm{Pic}_{X_{\psi(2)}/S}^0$  obtained as the composition of  $G^{[1]'} \rightarrow G$  with  $r_2: G \rightarrow P_{[\psi(2)]}$ . This map induces a morphism of pro-algebraic groups over  $k$ , again denoted by  $r'_2$ :

$$r'_2: \mathrm{Gr}(G^{[1]'}) \rightarrow \mathrm{Gr}(P_{[\psi(2)]})$$

We claim that this morphism factors through  $G_s^{[1]'} = \mathrm{Gr}_1(G^{[1]'}):$

$$(37) \quad \begin{array}{ccc} \mathrm{Gr}(G^{[1]'}) & \xrightarrow{r'_2} & \mathrm{Gr}(P_{[\psi(2)]}) \\ \downarrow & \nearrow \exists r'_{2,s} & \\ G_s^{[1]'} & & \end{array}$$

Let  $\mathcal{K}$  be the kernel of the morphism  $\mathrm{Gr}(G^{[1]'}) \rightarrow G_s^{[1]'}$ ; it is pro-smooth and connected. Let  $\varepsilon \in G^{[1]'}(S)$  be such that  $\varepsilon(s) \in G_s^{[1]'}$  is the unit element. In particular, the support of the torsion

module  $\mathcal{M}^{[1]'}$  at  $\varepsilon(s)$  has two irreducible components, which implies, according to Proposition 3.2.1 (1), that the  $\mathcal{O}_K$ -module  $\varepsilon^*\mathcal{M}^{[1]'} = H^1(X, \mathcal{L}_\varepsilon)$  has length at least 2 (here,  $\mathcal{L}_\varepsilon = (\text{id} \times \varepsilon)^*(\mathcal{L}|_{X \times G^{[1]'}})$ ). In particular, by Corollary 2.2.3, the restriction  $\mathcal{L}_\varepsilon|_{X_{\psi(2)}}$  is a power of  $\mathcal{I}|_{X_{\psi(2)}}$ . Since the invertible sheaf  $\mathcal{I}$  is of finite order and since  $k$  is algebraically closed, this implies that the induced map  $\mathcal{K} \rightarrow \text{Gr}(\text{P}_{[\psi(2)]})$  has finite image, in particular, it is trivial since  $\mathcal{K}$  is pro-smooth and connected. This fact ensures the existence of the factorization in (37).

In order to prove (i), let us denote by  $\ell_1$  the length of the  $\mathcal{O}_{G^{[1]'}, \xi'_1}$ -module  $\mathcal{M}_{\xi'_1}^{[1]'}$ . By definition of  $G^{[1]'}$ , we have  $\ell_1 \geq 1$ . Suppose  $\ell_1 \geq 2$ . Let  $\varepsilon \in G^{[1]'}(S)$  be a section of  $G^{[1]'}$  such that  $\varepsilon(s) \in \overline{\{\xi'_1\}} \subset G_s^{[1]'}$  and denote by  $\mathcal{L}_\varepsilon$  the associated rigidified invertible sheaf on  $X$ . According to Proposition 3.2.1 (1), the  $\mathcal{O}_K$ -module  $\varepsilon^*\mathcal{M}^{[1]'} \simeq H^1(X, \mathcal{L}_\varepsilon)$  is of length  $\geq \ell_1 \geq 2$ . Hence, by Corollary 2.2.3, we have  $\mathcal{L}_\varepsilon|_{X_{\psi(2)}} \simeq \mathcal{I}^i|_{X_{\psi(2)}}$  for a suitable integer  $i$ . Thus,  $r'_{2,s}(\overline{\{\xi'_1\}}) \subset \text{Gr}(\text{P}_{[\psi(2)]})$  consists of a single element because  $r'_{2,s}$  is continuous and the set  $\{\mathcal{I}^j : j \in \mathbb{Z}\}$  is finite. Using the fact that  $r'_{2,s}$  is a morphism of groups and that  $\overline{\{\xi'_1\}}$  is an irreducible component of  $G_s^{[1]'}$ , we deduce that the morphism  $r'_{2,s}$  and hence the map  $r'_2 : G^{[1]'}(S) \rightarrow \text{P}_{[\psi(2)]}(S) = \text{Pic}^0(X_{\psi(2)})$  has finite image. This contradicts the assertion on the infinity of the image of (36) proved above. Hence  $\ell_1 = 1$ , and this concludes the proof of (i).

Assertion (ii) is just a corollary of (i). In fact, for  $Y_1$  an irreducible component of  $H_s$ , let  $\xi'_1$  denote the generic point of  $G_s^{[1]'}$  lying above  $Y_1$ . Let  $x' \in \overline{\{\xi'_1\}} \subset G_s^{[1]'}$  be a closed point not contained in  $H_s^{[1]'}$ , and  $\varepsilon' : S \rightarrow G^{[1]'}$  a section lifting  $x'$ , which also gives a section  $\varepsilon : S \rightarrow G$  by composition with  $G^{[1]'} \rightarrow G$ . Let  $x = \varepsilon(s) \in H_s$ . Since  $x' \notin H_s^{[1]'}$ , and  $\ell(\mathcal{M}_{\xi'_1}^{[1]'}) = 1$  by assertion (i) of this Lemma, the  $\mathcal{O}_K$ -module  $\varepsilon'^*\mathcal{M}^{[1]'} = \varepsilon^*\mathcal{M}$  is of length 1 (see Proposition 3.2.1 (1)). According to Proposition 3.2.1 (2), this last condition implies that  $H$  is regular at  $x$ . Hence  $H$  is regular at the generic point of the irreducible component  $Y_1$  of  $H_s$  because  $Y_1$  contains  $x$ . Since this can be done for any generic point of  $H_s$ , one finds that  $H$  is normal by using Serre's criterion of normality (recall that the generic fibre  $H_K$  of  $H$  is regular, and the scheme  $H$ , being a divisor of a regular scheme, is Cohen-Macaulay).

For (iii), recall that the composed morphism  $G(S) \rightarrow \text{Pic}^0(X) \rightarrow J(S)$  is surjective (see § 1.1.2). Since  $G^{[1]'}$  is the dilatation of  $G$  along  $H_{s,\text{red}}$ , the surjectivity of the last map implies that the map  $G^{[1]'}(S) \rightarrow J^{[1]}(S)$  is also surjective. Since  $G^{[1]} \subset G^{[1]'}$  is an open subgroup, with non empty special fibre, and the abstract group  $G^{[1]'}(S)/G^{[1]}(S)$  is a finite group, according to [5], 9.2/6, the morphism  $\theta^{[1]}$  also induces a surjection  $G^{[1]}(S) \rightarrow J^{[1]}(S)$ . In particular, using the fact that the two  $S$ -group schemes  $G^{[1]}$  and  $J^{[1]}$  are smooth, we find that the morphism  $\theta^{[1]}$  is faithfully flat, and hence  $\ker(\theta^{[1]}) = H^{[1]}$  since both  $\ker(\theta^{[1]})$  and  $H^{[1]}$  are flat closed subgroup schemes of  $G^{[1]}$  having the same generic fibre.  $\square$

By abuse of notation, let us denote by  $r_2$  both the following composition of morphisms

$$r_2 : G^{[1]} \longrightarrow G \longrightarrow \text{P}_{[\psi(2)]}$$

and the induced morphism of pro-algebraic groups over  $k$ :  $\text{Gr}(G^{[1]}) \rightarrow \text{Gr}(\text{P}_{[\psi(2)]})$ . As we have seen in the proof of Lemma 3.4.2, diagram (37), (since  $G^{[1]} \subset G^{[1]'}$  is an open subgroup), the map  $r_2$  factors through the canonical surjection  $\text{Gr}(G^{[1]}) \rightarrow G_s^{[1]}$ :

$$\begin{array}{ccc} \text{Gr}(G^{[1]}) & \xrightarrow{r_2} & \text{Gr}(\text{P}_{[\psi(2)]}) \\ \downarrow & \nearrow \exists r_{2,s} & \\ G_s^{[1]} & & \end{array}$$

Next, define  $Z_1 := \ker(r_{2,s})_{\text{red}} \hookrightarrow G_s^{[1]}$ . Then the same argument used for  $Z$  in (31), in the case  $n = 1$ , implies that  $Z_1$  is a union of connected components of  $H_{s,\text{red}}^{[1]}$ .

Now we use constructions similar to those used in the comparison at the first level. Let  $G^{[2]}$  (respectively  $G^{[2]'}$ ) be the dilatation of  $G^{[1]}$  along the closed smooth subgroup  $Z_1$  of  $G_s^{[1]}$  (respectively along  $H_{s,\text{red}}^{[1]} \hookrightarrow G_s^{[1]}$ ), and let  $\alpha^{[1]}$  be the composed morphism  $G^{[2]} \rightarrow G^{[1]} \rightarrow G$ . According to [5], 3.2/3,  $G^{[2]}$  is a smooth  $S$ -group scheme, and we have an exact sequence:

$$0 \rightarrow G^{[2]}(S) \rightarrow G^{[1]}(S) \rightarrow P^{[1,\psi(2)]}(S) \rightarrow 0.$$

On the other hand, since  $H_s^{[1]}$  is the kernel of the morphism  $G_s^{[1]} \rightarrow J_s^{[1]}$ , according to the universal property of dilatations, we have an exact sequence of abstract groups

$$0 \rightarrow G^{[2]'}(S) \rightarrow G^{[1]}(S) \rightarrow J^{[1]}(S_1) \rightarrow 0.$$

Since  $Z_1 \subset H_{s,\text{red}}^{[1]}$  is an open subgroup scheme,  $G^{[2]} \subset G^{[2]'}$  is an open subgroup. Hence we obtain a morphism  $\alpha: P^{[1,\psi(2)]}(S) \rightarrow J^{[1]}(S_1)$  which renders the following diagram commutative:

$$(38) \quad \begin{array}{ccccc} G^{[1]}(S) & \longrightarrow & P^{[1]}(S) & \longrightarrow & P^{[1,\psi(2)]}(S) \\ & \searrow & \downarrow q^{[1]} & & \downarrow \alpha \\ & & J^{[1]}(S) & \longrightarrow & J^{[1]}(S_1) \end{array}$$

Hence we get a diagram with exact rows,

$$\begin{array}{ccccccc} P^{[1]}(S) & \hookrightarrow & \text{Pic}^0(X) & \xrightarrow{|_{X_1}} & \text{Pic}^0(X_1) & \longrightarrow & 0 \\ & \searrow \beta' & \downarrow q & \searrow \beta & & & \\ 0 & \longrightarrow & P^{[1,\psi(2)]}(S) & \longrightarrow & \text{Pic}^0(X_{\psi(2)}) & \longrightarrow & \text{Pic}^0(X_1) \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow q_2 & & \downarrow q_1 \\ J^{[1]}(S) & \hookrightarrow & J(S) & \longrightarrow & J(S_1) & \longrightarrow & J(S_1) \longrightarrow 0 \\ & & \downarrow \gamma & & \downarrow & & \downarrow \\ 0 & \longrightarrow & J^{[1]}(S_1) & \xrightarrow{\delta} & J(S_2) & \longrightarrow & J(S_1) \longrightarrow 0 \end{array}$$

This diagram (without the existence of  $q_2$ ) is seen to be commutative by combining the commutativity of diagrams (21), (34) and (38). In order to see that  $q_2$  exists, we need only show that  $\ker(\beta) \subset \ker(\gamma \circ q)$ . Since the upper horizontal sequence is the push-out of the “diagonal” exact sequence along  $\beta'$ , we have  $\ker(\beta) = \ker(\beta')$ . Hence to complete the proof, it is sufficient to recognize that the two maps  $P^{[1]}(S) \rightarrow J(S_2)$ , obtained, one following the path through  $\text{Pic}^0(X)$  and the other via  $P^{[1,\psi(2)]}(S)$ , coincide. This fact can be easily checked by diagram chasing. In this way, we have shown the existence of the morphism  $q_2$ .

Moreover, by Corollary 2.2.3, the kernel of  $q_2: \text{Pic}^0(X_{\psi(2)}) \rightarrow J(S_2)$  is generated by  $\mathcal{I}|_{X_{\psi(2)}} \in \text{Pic}^0(X_{\psi(2)})$ . In particular the morphism  $q$  in (24) induces a morphism  $q_2: \text{Gr}(\text{Pic}_{X_{\psi(2)}/S}^0) \rightarrow \text{Gr}_2(J)$  that is compatible with the morphism  $q_1$  in (35).

The general case can be done by induction on  $n$ , by using the same argument as before. Finally, we summarize our results in the following theorem:

**Theorem 3.4.3.** *Let the notation be as above. For any integer  $n \geq 1$ , the morphism of fppf-sheaves  $q: \text{Pic}_{X/S}^0 \rightarrow J$  induces a morphism of smooth  $k$ -group schemes*

$$q_n: \text{Gr}(\text{Pic}_{X_{\psi(n)/S}}^0) \rightarrow \text{Gr}_n(J)$$

*making the obvious diagram commute. Moreover, the morphism  $q_n$  defined above is an isogeny of algebraic  $k$ -groups, and the group of  $k$ -points of the kernel of  $q_n$  is given by*

$$\ker(q_n)(k) = \{\mathcal{I}^i|_{X_{\psi(n)}} : i \in \mathbf{Z}\} \subset \text{Gr}(\text{P}_{[\psi(n)]})(k) \simeq \text{Pic}^0(X_{\psi(n)}).$$

**Corollary 3.4.4.** *The morphism  $\mathbf{q}: \text{Pic}^0(X) \rightarrow \mathbf{J}(S)$  in (25) maps  $\mathbf{P}^{[\psi(n)]}(S)$  onto  $\mathbf{J}^{[n]}(S)$ , thus inducing an isogeny of connected quasi-algebraic groups*

$$\mathbf{q}_n: \text{Gr}(\text{P}_{[\psi(n)]}) \rightarrow \text{Gr}_n(J)$$

*whose kernel is generated by the element  $\mathcal{I}|_{X_{\psi(n)}} \in \text{Gr}(\text{P}_{[\psi(n)]})(k) = \text{Pic}^0(X_{\psi(n)})$ . Furthermore the kernel of  $\mathbf{q}$  is isomorphic to  $\mathbb{Z}/d\mathbb{Z}$ .*

*Proof.* According to the previous theorem, we have  $\mathbf{q}(\mathbf{P}^{[\psi(n)]}(S)) \subset \mathbf{J}^{[n]}(S)$ , hence  $\mathbf{q}$  induces the isogeny  $\mathbf{q}_n$  with properties as stated in the corollary. In order to finish the proof, we need only establish that the last inclusion is in fact an equality. To see this, we consider the quotient of  $\mathbf{J}^{[n]}(S)$  by  $\mathbf{q}(\mathbf{P}^{[\psi(n)]}(S))$ . By applying the snake Lemma to the following diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{P}^{[\psi(n)]}(S) & \longrightarrow & \text{Pic}^0(X) & \longrightarrow & \text{Gr}(\text{P}_{[\psi(n)]}) \longrightarrow 0 \\ & & \downarrow & & \downarrow q & & \downarrow q_n \\ 0 & \longrightarrow & \mathbf{J}^{[n]}(S) & \longrightarrow & \mathbf{J}(S) & \longrightarrow & \text{Gr}_n(J) \longrightarrow 0 \end{array}$$

we find that the quotient  $\mathbf{J}^{[n]}(S)$  by  $\mathbf{P}^{[\psi(n)]}(S)$  is a finite quasi-algebraic group. Since  $\mathbf{J}^{[n]}/S$  has connected fibres, the pro-algebraic group  $\mathbf{J}^{[n]}(S)$  is connected, hence the cokernel of the left vertical arrow is necessarily trivial. Thus  $\mathbf{q}(\mathbf{P}^{[\psi(n)]}(S)) = \mathbf{J}^{[n]}(S)$ . The kernel of  $\mathbf{q}$  is cyclic of order  $d$  because the kernel of any  $\mathbf{q}_n$  is a constant finite group and the kernel of  $q$  in (11) is isomorphic to  $\mathbb{Z}/d\mathbb{Z}$  and is generated by  $\mathcal{I}$  ([16], Théorème 6.4.1 (3)).  $\square$

**Corollary 3.4.5.** *Let notations be as above. The following conditions are equivalent:*

- (1) *The torsor  $X_K$  is tamely ramified;*
- (2) *The Picard functor  $\text{Pic}_{X/S}^0$  is representable, and the canonical map  $\text{Pic}_{X/S}^0 \rightarrow J$  is étale.*
- (3) *The extension of Serre pro-algebraic groups associated to  $X/S$*

$$0 \rightarrow \mathbb{Z}/d\mathbb{Z} \rightarrow \text{Pic}^0(X) \rightarrow \mathbf{J}(S) \rightarrow 0$$

*lies in the subgroup  $\text{Ext}^1(\text{Gr}_1(J), \mathbb{Z}/d\mathbb{Z}) \subset \text{Ext}^1(\mathbf{J}(S), \mathbb{Z}/d\mathbb{Z})$ .*

*Proof.* The equivalence between (1) and (2) follows from Proposition 5.2 of [16] and from Corollary 2.3.3. To see (1)  $\Leftrightarrow$  (3), suppose first that  $X_K$  is tamely ramified, namely  $\mathcal{I}|_{X_1}$  is of order  $d$ ; we then have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/d\mathbb{Z} & \longrightarrow & \mathbf{Pic}^0(X) & \xrightarrow{q} & J(S) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z}/d\mathbb{Z} & \longrightarrow & \mathbf{Gr}(\mathbf{P}_{[1]}) & \xrightarrow{q_1} & \mathbf{Gr}_1(J) \longrightarrow 0 \end{array}$$

In particular, we get (3). Conversely, if condition (3) holds, the morphism  $q$  induces an isomorphism between  $q^{-1}(J^{[1]}(S))$  and  $J^{[1]}(S) = \ker(J(S) \rightarrow \mathbf{Gr}_1(J))$ . In particular, Serre pro-algebraic group  $q^{-1}(J^{[1]}(S))$  is connected. On the other hand,  $q^{-1}(J^{[1]}(S))$  contains the subgroup  $\mathbf{Gr}(\mathbf{P}_{[1]})$  of index  $d/d_1$  with  $d_1$  the order of  $\mathcal{I}|_{X_1}$ . As a result, we find  $q^{-1}(J^{[1]}(S)) = \mathbf{Gr}(\mathbf{P}_{[1]})$  by the connectedness of  $J^{[1]}(S)$ . Therefore,  $d = d_1$ , and  $\mathcal{I}|_{X_1}$  is of order  $d$ , hence  $X_K$  is tamely ramified (Corollary 2.3.3).  $\square$

**Remark 3.4.6.** Let  $G^{[0]'} = G$ ,  $\theta^{[0]'} = \theta: G \rightarrow J$ . Define by induction, for each integer  $n \geq 0$ , a smooth  $S$ -group scheme  $G^{[n]}'$ , and a faithfully flat morphism of  $S$ -group schemes  $\theta^{[n]'}: G^{[n]}' \rightarrow J^{[n]}$  in the following way: let  $n \geq 1$  and suppose we have constructed  $G^{[n-1]}'$  and  $\theta^{[n-1]'}: G^{[n-1]}' \rightarrow J^{[n-1]}$ . Since  $\theta^{[n-1]}'$  is faithfully flat, its kernel  $H^{[n-1]}' = \ker(\theta^{[n-1]'})$  is an  $S$ -group scheme flat over  $S$ . Then we define  $G^{[n]}'$  as the dilatation of  $G^{[n-1]}'$  along  $H_{s,\text{red}}^{[n-1]}' \subset G_s^{[n-1]}'$ .  $G^{[n]}'$  is a smooth  $S$ -group scheme ([5], 3.2/3). By the universal property of dilatations ([5] 3.2/1), the morphism  $\theta^{[n-1]}'$  induces a morphism of  $S$ -group scheme  $\theta^{[n]'}: G^{[n]}' \rightarrow J^{[n]}$ , which is also faithfully flat (proof as for Lemma 3.4.2 (iii)). Now let  $H^{[n]}' := \ker(\theta^{[n]'})$ . Then the proof of Theorem 3.4.3 (especially of Lemma 3.4.2 (ii)) shows that the scheme  $H^{[n]}'$  is normal. Moreover, we can verify that the scheme  $H^{[n]}'$  is smooth over  $S$  for sufficiently large  $n$ .

## 4 Shafarevich's pairing

As stated in the introduction, this section gives a new construction of a homomorphism as in (5) via the rigidified Picard functor. We will prove that it coincides with the classical Shafarevich's duality if  $d$  is prime to  $p$ , and more generally, for all  $d$  in the mixed characteristic case. In Section 5, we will use these constructions to study the morphism  $\Phi_d$  in (5).

### 4.1 The component group of a torus

One of the key facts in the construction of Shafarevich's duality is the pro-algebraic structure of the cohomology group  $H_{\text{fl}}^1(K, \mu_n)$ , where  $\mu_n$  denotes the finite subgroup scheme of  $n$ -th roots of unity in the multiplicative group  $\mathbb{G}_{m,K}$ .

Recall that the Néron model  $T$  of a torus  $T_K$  is locally of finite type, but, in general, not of finite type over  $S = \text{Spec}(\mathcal{O}_K)$ . It is of finite type if and only if it does not contains split tori (cf. [5], 10.2/1). Let  $\Lambda_K$  denote the group of characters of  $T_K$ . If  $T_K$  has no non trivial split quotient, *i.e.*, if  $\Lambda_K(K) = 0$ , then  $T$  is of finite type.

**Lemma 4.1.1.** *Let  $f: T_{1,K} \rightarrow T_{2,K}$  be an isogeny of tori with kernel a finite group scheme  $F_K$ . The group  $H_{\text{fl}}^1(K, F_K) = T_{2,K}(K)/T_{1,K}(K)$  has a canonical pro-algebraic structure.*

*Proof.* (Cf. [1], 4.3.) Let  $X_{i,K}$  (respectively  $T_i$ ) be the character group (respectively the Néron model) of  $T_{i,K}$ ,  $i = 1, 2$ . Denote by  $T_{1,K}^{(d)}$  the torus (*déployé*) whose group of characters is the

constant free group  $\Lambda_{1,K}(K)$ , and similarly for  $T_{2,K}^{(d)}$ . They are split tori with the same component group, say  $\mathbb{Z}^r$ . Furthermore the isogeny  $f$  induces an isogeny  $f^{(d)}: T_{1,K}^{(d)} \rightarrow T_{2,K}^{(d)}$  that is injective on component groups. The torus  $T'_{i,K}$ , defined as the kernel of the quotient map  $T_{i,K} \rightarrow T_{i,K}^{(d)}$ , admits a Néron model of finite type because its group of characters is  $\Lambda'_{i,K} = \Lambda_{i,K}/\Lambda_{i,K}(K)$ . Hence, using the exact sequences

$$\pi_0(T'_i) \longrightarrow \pi_0(T_i) \longrightarrow \pi_0(T_i^{(d)}) \longrightarrow 0, \quad i = 1, 2,$$

one sees that the kernel and the cokernel of the homomorphism  $\pi_0(T_1) \rightarrow \pi_0(T_2)$  are finite groups. The identity components of the Néron models  $T_i$  are smooth group schemes of finite type ([5], 10.1). Hence their perfect Greenberg realizations are pro-algebraic groups. Let us denote by  $\mathbf{P}$  the cokernel of the map  $\mathbf{Gr}(T_1^0) \rightarrow \mathbf{Gr}(T_2^0)$ . Now, the cokernel of the map  $\mathbf{Gr}(T_1) \rightarrow \mathbf{Gr}(T_2)$  is an extension of the finite group  $\pi_0(T_2)/\pi_0(T_1)$  by the quotient of  $\mathbf{P}$  by a finite constant group. Hence it is a Serre pro-algebraic group that will be denoted by  $\mathbf{H}^1(K, F_K)$  since its group of  $k$ -points is  $H_{\text{fl}}^1(K, F_K)$ .  $\square$

For our later work we will also need the following result:

**Lemma 4.1.2.** *Let  $0 \rightarrow T \rightarrow G \rightarrow A^0 \rightarrow 0$  be an exact sequence of smooth group schemes over  $S$ , where  $T$  is the Néron model of a torus and  $A^0$  is the identity component of the Néron model of an abelian variety. This exact sequence induces a homomorphism of profinite groups  $\pi_1(\mathbf{Gr}(A^0)) \rightarrow \pi_0(T)_{\text{tor}}$  where the index  $\text{tor}$  indicates the torsion subgroup.*

*Proof.* If  $T$  is of finite type, the proof is immediate. Indeed  $\pi_0(T)_{\text{tor}} = \pi_0(T) = \pi_0(\mathbf{Gr}(T))$  is finite and, on applying the perfect Greenberg functor to the above sequence, we get an extension

$$0 \rightarrow \mathbf{Gr}(T) \rightarrow \mathbf{Gr}(G) \rightarrow \mathbf{Gr}(A^0) \rightarrow 0.$$

The desired map then follows from the long exact sequence of the  $\pi_i$ 's (see § 1.2).

Suppose that  $T$  is locally of finite type. Since  $k$  is algebraically closed,  $\pi_0(T) = \pi_0(T)_{\text{tor}} \oplus \pi_0(T)_{\text{fr}}$  with  $\pi_0(T)_{\text{fr}}$  torsion-free. Let  $T^{\text{ft}}$  be the maximal subgroup of  $T$  whose component group is finite. In particular,  $T^{\text{ft}}$  contains the identity component  $T^0$  and  $\pi_0(T^{\text{ft}}) = \pi_0(T)_{\text{tor}}$ . The group  $G$  is obtained as the push-out along the inclusion map  $T^{\text{ft}} \rightarrow T$  of a unique (up to isomorphism) extension  $0 \rightarrow T^{\text{ft}} \rightarrow G^{\text{ft}} \rightarrow A^0 \rightarrow 0$  because the quotient  $T/T^{\text{ft}}$  satisfies the hypothesis in [8], § 5.7, 5.5. Hence we can proceed as above, with  $G^{\text{ft}}$  in place of  $G$ , obtaining a map  $\pi_1(\mathbf{Gr}(A)) \rightarrow \pi_0(\mathbf{Gr}(T^{\text{ft}})) = \pi_0(T)_{\text{tor}}$ .  $\square$

## 4.2 Bégueri's construction

In this section we assume that  $K$  has characteristic 0. Given  $K$ -schemes  $Z_K$  and  $U_K$ , let us denote by  $Z_{U_K}$  the fibred product  $Z_K \times_K U_K$ , viewed as a scheme over  $U_K$ .

Let  $X_K$  be a  $K$ -torsor under  $A_K$  and let  $n$  be a positive integer such that  $nX_K$  is trivial; the order of  $X_K$  is the minimum among such integers. The torsor  $X_K$  corresponds to an  $n$ -torsion element in  $H_{\text{fl}}^1(K, A_K) = \text{Ext}^1(\mathbb{Z}, A_K)$  and hence to an extension of group schemes over  $K$ :

$$(39) \quad 0 \rightarrow A_K \rightarrow B_K \rightarrow \mathbb{Z} \rightarrow 0,$$

which is the pull-back along  $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$  of a, not unique, extension

$$(40) \quad \eta: 0 \rightarrow A_K \xrightarrow{\alpha} E_K \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0;$$



the fibre at  $1 \in \mathbb{Z}/n\mathbb{Z}$  is precisely  $X_K$ . Let us denote by

$$(41) \quad \eta_n: 0 \rightarrow {}_nA_K \rightarrow {}_nE_K \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0$$

the sequence of  $n$ -torsion subgroups. Consider also the exact sequence

$$(42) \quad 0 \rightarrow \mu_n \rightarrow V_{nE_K}^* \rightarrow \underline{\mathrm{Ext}}^1(E_K, \mathbb{G}_m)_{nE_K} \xrightarrow{\tau_E} \underline{\mathrm{Ext}}^1(E_K, \mathbb{G}_m) = A'_K \rightarrow 0$$

(cf. [1], 2.3.2) where, as in § 1.1, we denote by  $V_{nE_K}^*$  the torus  $\mathfrak{R}_{nE_K/K}(\mathbb{G}_{m,nE_K})$  representing the Weil restriction functor that associates to a  $K$ -scheme  $S'$  the group  $\mathbb{G}_{m,K}(S' \times_K nE_K)$  ([5], 7.6). Observe that

$$\mu_n = \underline{\mathrm{Hom}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) = \underline{\mathrm{Hom}}(E_K, \mathbb{G}_m).$$

The second map in (42) sends a homomorphism  $f: E_K \rightarrow \mathbb{G}_{m,K}$  to its restriction to  ${}_nE_K$ , while the third arrow sends  $g \in \mathbb{G}_{m,K}({}_nE_K)$  to (the isomorphism class of) the trivial extension endowed with the section  $g$ , and the map  $\tau_E$  forgets the rigidification along  ${}_nE_K$ .

We now describe Bégueri's construction of Shafarevich's duality following [1]. Let  $F_K$  be a finite  $K$ -group scheme and  $F_K^D$  its Cartier dual. There is a short exact sequence (cf. [1], 2.2.1)

$$(43) \quad 0 \rightarrow F_K^D \rightarrow V_{F_K}^* \rightarrow \underline{\mathrm{Ext}}^1(F_K, \mathbb{G}_m)_{F_K} \rightarrow 0,$$

where the second map forgets the group structure and the third map associates to each  $f \in \mathbb{G}_{m,K}(F_K)$  the trivial extension endowed with the rigidification induced by  $f$ . We also recall the following exact sequence (cf. [1], 2.3.1)

$$(44) \quad 0 \rightarrow V_{nA_K}^* \rightarrow \underline{\mathrm{Ext}}^1(A_K, \mathbb{G}_m)_{nA_K} \xrightarrow{\tau_A} A'_K \rightarrow 0.$$

In [1], 8.2.2 Bégueri first constructs a map

$$\Gamma: H_{\mathrm{fl}}^1(K, {}_nA_K) \rightarrow \mathrm{Ext}^1(\mathbf{Gr}(A'), \mathbf{H}^1(K, \mu_n))$$

as follows: any element in  $H_{\mathrm{fl}}^1(K, {}_nA_K)$  corresponds to a sequence  $\eta_n$  as in (41). Consider now the diagram

$$\begin{array}{ccccc} \mu_n & \xlongequal{\quad} & \mu_n & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ V_{\mathbb{Z}/n\mathbb{Z}}^* & \xrightarrow{\quad} & V_{nE_K}^* & \xrightarrow{\quad} & V_{nA_K}^* \\ \downarrow v_1 & & \downarrow v_2 & & \downarrow v_3 \\ \underline{\mathrm{Ext}}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)_{\mathbb{Z}/n\mathbb{Z}} & \longrightarrow & \underline{\mathrm{Ext}}^1(E_K, \mathbb{G}_m)_{nE_K} & \longrightarrow & \underline{\mathrm{Ext}}^1(A_K, \mathbb{G}_m)_{nA_K} \\ \downarrow & & \downarrow \tau_E & & \downarrow \tau_A \\ 0 & & A'_K & \xlongequal{\quad} & A'_K \end{array}$$

where the rows are complexes and the vertical sequences are those in (43), for  $F_K = \mathbb{Z}/n\mathbb{Z}$ , (42), (44), respectively. Since  $K$  has characteristic 0, the second row consists of tori, while the third row consists of semi-abelian varieties. Hence they all admit Néron models. On passing to the perfection of the Greenberg realization of the Néron models and considering the cokernels of the maps induced by  $v_1, v_2, v_3$ , one gets a complex of pro-algebraic groups (cf. Lemma 4.1.1)

$$(45) \quad 0 \rightarrow \mathbf{H}^1(K, \mu_n) \rightarrow \mathrm{Ext}^1(E_K, \mathbb{G}_m) \rightarrow \mathbf{Gr}(A') \rightarrow 0;$$

this is indeed an exact sequence because on  $k$ -points it induces the exact sequence

$$0 \rightarrow H_{\text{fl}}^1(K, \mu_n) = \text{Ext}^1(\mathbb{Z}/d\mathbb{Z}, \mathbb{G}_m) \rightarrow \text{Ext}^1(E_K, \mathbb{G}_m) \rightarrow A'(\mathcal{O}_K) = \text{Ext}^1(A_K, \mathbb{G}_m) \rightarrow 0.$$

We have thus associated with (41) an extension of  $\mathbf{Gr}(A')$  by  $\mathbf{H}^1(K, \mu_n)$ : this is the image of (41) via  $\Gamma$ .

The homomorphism

$$\psi_n: H_{\text{fl}}^1(K, {}_nA_K) \longrightarrow \text{Ext}^1(\mathbf{Gr}(A'^0), \mathbb{Z}/n\mathbb{Z})$$

in [1], 8.2.3, is then obtained by applying first  $\Gamma$ , then the pull-back along  $\mathbf{Gr}(A'^0) \rightarrow \mathbf{Gr}(A')$  and, finally, the push-out along  $\mathbf{H}^1(K, \mu_n) \rightarrow \pi_0(\mathbf{H}^1(K, \mu_n)) = \mathbb{Z}/n\mathbb{Z}$ . Let us denote by

$$(46) \quad \psi_n(\eta_n): \quad 0 \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow W(X_K) \rightarrow \mathbf{Gr}(A'^0) \rightarrow 0$$

the image of (41) via  $\psi_n$ . Recall now that (cf. [18], 5.4)

$$(47) \quad \text{Ext}(\mathbf{Gr}(A'^0), \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\pi_1(\mathbf{Gr}(A'^0)), \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\pi_1(\mathbf{Gr}(A')), \mathbb{Q}/\mathbb{Z}).$$

In terms of homomorphisms of profinite groups, the extension (46) then corresponds to a map

$$(48) \quad u^\tau = u_{X_K}^\tau: \pi_1(\mathbf{Gr}(A'^0)) \longrightarrow \pi_0(\mathbf{H}^1(K, \mu_n)) = \mathbb{Z}/n\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$$

deduced from (45) (or equivalently, from the pull-back of (45) along  $\mathbf{Gr}(A'^0) \rightarrow \mathbf{Gr}(A')$ ) via the long exact sequence of  $\pi_i$ 's.

**Lemma 4.2.1** ([1], 8.2.3). *Let notations be as above.*

- i) *The extension  $\psi_n(\eta_n)$  in (46) depends only on the sequence (39), i.e., on the torsor  $X_K$ ;*
- ii) *the formation of  $\psi_n(\eta_n)$  behaves well with respect to the inclusions  $H_{\text{fl}}^1(K, {}_nA_K) \rightarrow H_{\text{fl}}^1(K, {}_{n'}A_K)$ , and  $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n'\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$  for  $n|n'$ ;*
- iii) *the above construction provides the duality in (7).*

Hence we can deduce that

**Lemma 4.2.2.** *Shafarevich's duality in (7) maps the class of the torsor  $X_K$  to the extension corresponding via (47) to the homomorphism  $u_{X_K}^\tau$  in (48).*

#### 4.2.1 An alternative construction of $\psi_n(\eta_n)$ in (46) (in view of further applications)

The kernel of  $\tau_E$  in (42) is a torus, which, for brevity, we denote by  $T_K^\tau$ . Let  $T^\tau$  be its Néron model over  $S$ . We have an exact sequence

$$0 \rightarrow T_K^\tau \rightarrow \underline{\text{Ext}}^1(E_K, \mathbb{G}_m)_{nE_K} \xrightarrow{\tau_E} A'_K \rightarrow 0$$

which extends to an exact sequence of Néron models

$$(49) \quad 0 \rightarrow T^\tau \rightarrow j_* \underline{\text{Ext}}^1(E_K, \mathbb{G}_m)_{nE_K} \rightarrow A' \rightarrow 0.$$

On applying the perfection of the Greenberg functor we get an exact sequence

$$(50) \quad 0 \rightarrow \mathbf{Gr}(T^\tau) \rightarrow \mathbf{Gr}(j_* \underline{\text{Ext}}^1(E_K, \mathbb{G}_m)_{nE_K}) \xrightarrow{\tau} \mathbf{Gr}(A') \rightarrow 0$$

where the first two groups are not pro-algebraic in general, because they are projective limits of perfect schemes not necessarily of finite type. Nevertheless, on applying the perfection of the Greenberg functor to the morphism of Néron models  $j_* V_{nE_K}^* \rightarrow T^\tau$  we get a homomorphism whose cokernel is a pro-algebraic group (cf. Lemma 4.1.1) and whose group of  $k$ -points is  $H^1(K, \mu_n)$ ; we will write

$$\mathbf{Gr}(j_* V_{nE_K}^*) \rightarrow \mathbf{Gr}(T^\tau) \xrightarrow{h^\tau} \mathbf{H}^1(K, \mu_n) \rightarrow 0.$$

Now take the push-out of (50) along  $h^\tau$ ; by construction, the resulting exact sequence is the one in (45), *i.e.*, the image of (41) via  $\Gamma$ . Hence, if one considers the pull-back of (50) along  $\mathbf{Gr}(A^0) \rightarrow \mathbf{Gr}(A')$ ,

$$(51) \quad 0 \rightarrow \mathbf{Gr}(T^\tau) \rightarrow U \rightarrow \mathbf{Gr}(A^0) \rightarrow 0,$$

and then the push-out of (51) along the composition of maps

$$\mathbf{Gr}(T^\tau) \xrightarrow{h^\tau} \mathbf{H}^1(K, \mu_n) \rightarrow \pi_0(\mathbf{H}^1(K, \mu_n)) = \mathbb{Z}/n\mathbb{Z},$$

one gets the extension  $\psi_n(\eta_n)$  in (46), *i.e.*, the image of  $X_K$  via Shafarevich's duality.

Thanks to this new description of Shafarevich's map, we can characterize the map  $u^\tau$  in (48) as follows:

First consider the pull-back of (49) along  $A^0 \rightarrow A'$ . As we have seen in the proof of Lemma 4.1.2, this extension is the push-out of an extension

$$0 \rightarrow T^{\tau, \text{ft}} \rightarrow G \rightarrow A^0 \rightarrow 0$$

where  $T^{\tau, \text{ft}}$  is the maximal subgroup scheme of finite type of  $T^\tau$ . The pull-back of the sequence in (50) along  $\mathbf{Gr}(A^0) \rightarrow \mathbf{Gr}(A')$  is then isomorphic to the push-out of

$$(52) \quad 0 \rightarrow \mathbf{Gr}(T^{\tau, \text{ft}}) \rightarrow \mathbf{Gr}(G) \rightarrow \mathbf{Gr}(A^0) \rightarrow 0$$

along the composition of maps  $h^{\tau, \text{ft}}: \mathbf{Gr}(T^{\tau, \text{ft}}) \rightarrow \mathbf{Gr}(T^\tau) \xrightarrow{h^\tau} \mathbf{H}^1(K, \mu_n)$ . Hence

$$(53) \quad u_{X_K}^\tau = \pi_0(h^{\tau, \text{ft}}) \circ w$$

where the homomorphism  $w: \pi_1(\mathbf{Gr}(A')) \rightarrow \pi_0(\mathbf{Gr}(T^{\tau, \text{ft}}))$  is deduced from (52) via the long exact sequence of the  $\pi_i$ 's.

### 4.3 An alternative construction using rigidifiers

Let  $X_K$  be a torsor under an abelian variety  $A_K$ . We will see in this section how the homomorphism  $u^\tau$  in (48) (and in (53)) can be constructed using a rigidifier  $x_K$  of the relative Picard functor  $\text{Pic}_{X_K/K}$ . Observe that any closed point of  $X_K$  provides a rigidifier of  $\text{Pic}_{X_K/K}$ .

**Lemma 4.3.1.** *Let  $X_K$  be a torsor under an abelian variety  $A_K$ , of order  $d$ . Let  $d'$  be the separable index of  $X_K$ , *i.e.*, the greatest common divisor of the degrees of its finite separable splitting extensions. Then  $d|d'$  and they have the same prime factors. If  $A_K$  is an elliptic curve, then  $d = d'$  and the index is indeed the degree of a minimal separable splitting extension.*

*Proof.* Using the restriction and corestriction maps, we find that  $nX_K = 0$  if  $X_K$  becomes trivial over a finite separable extension  $K'/K$  of degree  $n$ . Hence  $d|n$ . Suppose now that we are given separable extensions  $K \subseteq L \subseteq L'$  with  $(d, [L' : L]) = 1$  and  $X_{L'} = 0$ . Then  $[L' : L] \cdot X_L = 0$  in  $H_{\text{fl}}^1(L, A_L)$ . However the order of  $X_L$  in  $H_{\text{fl}}^1(L, A_L)$  divides  $d$  and thus  $X_L = 0$ . Hence  $d, d'$  have the same prime factors.

For the latter assertion on elliptic curves see Lemma 2.1.2.  $\square$

**Remark 4.3.2.** Let  $x_K = \text{Spec}(K')$  with  $K'/K$  a finite separable extension of degree  $n$ . Then the torus  $V_{x_K}^*$  is the Weil restriction  $\mathfrak{R}_{K'/K}(\mathbb{G}_{m,K'})$ , it has component group isomorphic to  $\mathbb{Z}$  and the closed immersion  $\mathbb{G}_{m,K} \rightarrow V_{x_K}^*$  (i.e., the inclusion  $K^* \subset K'^*$  on  $K$ -sections) induces the  $n$ -multiplication  $n: \mathbb{Z} \rightarrow \mathbb{Z}$  on component groups of Néron models over  $S$ .

#### 4.3.1 An alternative construction of Shafarevich's duality

The main idea here is to use in (42) a rigidificator  $x_K$  of  $\text{Pic}_{X_K/K}$  in place of  ${}_nE_K$ . The advantage is that the new construction works even for  $K$  of positive characteristic; in this case we choose  $x_K$  étale so that  $V_{x_K}^* = \mathfrak{R}_{x_K/K}(\mathbb{G}_{m,x_K})$  is still a torus.

Observe that a rigidificator  $x_K$  is a closed subscheme of  $E_K$  and the homomorphism

$$\mu_n = \underline{\text{Hom}}(E_K, \mathbb{G}_m) \rightarrow V_{x_K}^*$$

is still a closed immersion. Indeed any homomorphism  $f: E_K \rightarrow \mathbb{G}_m$  factors through  $\rho: E_K \rightarrow \mathbb{Z}/n\mathbb{Z}$  and if  $f|_{x_K} = 0$  then  $f|_{X_K} = 0$  because  $x_K$  is a rigidificator. However  $X_K$  is the fibre at 1 of  $\rho$  and hence also  $f = 0$ . We then have an exact sequence

$$(54) \quad 0 \rightarrow \mu_n = \underline{\text{Hom}}(E_K, \mathbb{G}_m) \rightarrow V_{x_K}^* \rightarrow \underline{\text{Ext}}^1(E_K, \mathbb{G}_m)_{x_K} \rightarrow A'_K \rightarrow 0.$$

More generally, we will say that a finite étale subscheme  $Z_K$  of  $E_K$  satisfies property  $(*)$  if

$$(*) \quad \text{the canonical map } \mu_n = \underline{\text{Hom}}(E_K, \mathbb{G}_m) \rightarrow V_{Z_K}^* \text{ is a closed immersion.}$$

For any such étale subscheme  $x_K$  we can construct an exact sequence as in (54).

Denote by  $T_K^x$  the torus  $V_{x_K}^*/\mu_n$  and omit the exponent  $x$  if the rigidificator  $x_K$  is fixed. The sequence (54) induces an exact sequence

$$(55) \quad 0 \rightarrow T_K \rightarrow \underline{\text{Ext}}^1(E_K, \mathbb{G}_m)_{x_K} \rightarrow A'_K \rightarrow 0,$$

and hence an exact sequence (cf. Lemma 4.1.2/proof)

$$(56) \quad 0 \rightarrow T^{\text{ft}} \rightarrow G_1 \rightarrow A'^0 \rightarrow 0,$$

where  $T^{\text{ft}}$  is the maximal subgroup of finite type of the Néron model  $T$  of  $T_K$ . Now consider the cokernel

$$(57) \quad \mathbf{Gr}(j_* V_{x_K}^*) \xrightarrow{g^x} \mathbf{Gr}(T) \xrightarrow{h} \mathbf{H}^1(K, \mu_n) \rightarrow 0$$

of the homomorphism between the perfect Greenberg realizations of the Néron models of  $V_{x_K}^*$  and  $T_K$ ; by Lemma 4.1.1 it is a pro-algebraic group whose group of  $k$ -points is  $\mathbf{H}_{\mathfrak{h}}^1(K, \mu_n)$ .

**Lemma 4.3.3.** *The pro-algebraic group  $\mathbf{H}^1(K, \mu_n)$  in (57) does not depend on the étale finite subscheme  $x_K$  chosen to construct it. In particular it coincides with that of (42).*

*Proof.* Let  $x_K \subset y_K$  be finite étale subschemes of  $E_K$  satisfying  $(*)$ . We have canonical morphisms  $f^V: V_{y_K}^* \rightarrow V_{x_K}^*$ ,  $f^T: T_K^y \rightarrow T_K^x$ , such that  $f^T \circ g^y = g^x \circ f^V$ . Hence the pro-algebraic group constructed in (57) for  $x_K$  is canonically isomorphic to the one constructed via  $y_K$ .  $\square$

In order to provide a more useful description of the map  $u^\tau$  in (48), consider the perfect Greenberg realization of (56)

$$(58) \quad 0 \rightarrow \mathbf{Gr}(T^{\text{ft}}) \rightarrow \mathbf{Gr}(G_1) \rightarrow \mathbf{Gr}(A'^0) \rightarrow 0,$$

and then its push-out along the composition of maps

$$(59) \quad h^{\text{ft}}: \mathbf{Gr}(T^{\text{ft}}) \rightarrow \mathbf{Gr}(T) \xrightarrow{h} \mathbf{H}^1(K, \mu_n).$$

We obtain an exact sequence

$$\zeta: 0 \rightarrow \mathbf{H}^1(K, \mu_n) \rightarrow W' \rightarrow \mathbf{Gr}(A'^0) \rightarrow 0$$

and hence a homomorphism

$$u_{X_K} = u: \pi_1(\mathbf{Gr}(A')) \rightarrow \pi_0(\mathbf{H}^1(K, \mu_n)) = \mathbb{Z}/n\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$$

such that

$$(60) \quad u = \pi_0(h^{\text{ft}}) \circ u^{\text{ft}},$$

where  $u^{\text{ft}}: \pi_1(\mathbf{Gr}(A')) \rightarrow \pi_0(\mathbf{Gr}(T^{\text{ft}})) = \pi_0(T)_{\text{tor}}$  is deduced from the long exact sequence of the  $\pi_i$ 's of (58).

**Proposition 4.3.4.** *The association  $X_K \mapsto u_{X_K}$  provides a homomorphism*

$$\Xi: \mathbf{H}^1(K, A_K) \rightarrow \text{Hom}(\pi_1(\mathbf{Gr}(A')), \mathbb{Q}/\mathbb{Z}).$$

If  $\text{char}(K) = 0$  the homomorphism  $u_{X_K}$  in (60) coincides with the homomorphism  $u_{X_K}^\tau$  in (48). In particular, the homomorphism  $\Xi$  is, up to the identifications in (47), Shafarevich's duality in (7).

*Proof.* We start by showing that, once  $X_K$  has been fixed, the construction of  $u: \pi_1(\mathbf{Gr}(A')) \rightarrow \mathbb{Q}/\mathbb{Z}$  in (60) does not depend on the choices of  $x_K$ ,  $n$  and  $\eta \in \text{Ext}^1(\mathbb{Z}/n\mathbb{Z}, A_K)$  above  $X_K$ .

First we see that  $u$  does not depend on the étale finite closed subscheme  $x_K$  of  $E_K$  satisfying (\*). Let  $x_K \subset y_K$  be two étale subschemes of  $E_K$  satisfying (\*). Denote by  $T_K^x, h^x, h^{\text{ft},x}, u'^x, u^x$  respectively the torus in (55), the maps in (57), (59) and (60) for  $x_K$ , and similarly for  $y_K$ . The canonical morphism of tori  $T_K^y \rightarrow T_K^x$  induces a morphism  $\beta: T^{y,\text{ft}} \rightarrow T^{x,\text{ft}}$  between the maximal subgroups of finite type of the Néron models. Denote by  $\beta': \mathbf{Gr}(T^{y,\text{ft}}) \rightarrow \mathbf{Gr}(T^{x,\text{ft}})$  the corresponding map on perfect Greenberg realizations. One then has  $\beta' \circ h^{x,\text{ft}} = h^{y,\text{ft}}$  and  $\pi_0(h^{y,\text{ft}}) = \pi_0(h^{x,\text{ft}}) \circ \pi_0(\beta')$ . Furthermore the sequence (58) for  $x_K$  is the push-out along  $\beta'$  of the sequence (58) for  $y_K$ . Hence  $u^{x,\text{ft}} = \pi_0(\beta') \circ u^{y,\text{ft}}$ . We conclude then that

$$(61) \quad u^x = \pi_0(h^{x,\text{ft}}) \circ u^{x,\text{ft}} = \pi_0(h^{x,\text{ft}}) \circ \pi_0(\beta') \circ u^{y,\text{ft}} = \pi_0(h^{y,\text{ft}}) \circ u^{y,\text{ft}} = u^y.$$

Let now  $n, \hat{n}$  be positive integers such that  $n \cdot X_K = 0$  and  $n|\hat{n}$ . We can consider the pull-back  $\hat{\eta}$  of  $\eta$  in (40) along the projection  $\mathbb{Z}/\hat{n}\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ . If we proceed with  $\hat{\eta}$  as we have done for  $\eta$ , we get a map  $\hat{u}: \pi_1(\mathbf{Gr}(A')) \rightarrow \mathbb{Q}/\mathbb{Z}$ . Observe that the 2-fold extension (54) for  $\hat{\eta}$  is the push-out along  $\mu_n \rightarrow \mu_{\hat{n}}$  of (54) and that the map  $\pi_0(\mathbf{H}^1(K, \mu_n)) \rightarrow \pi_0(\mathbf{H}^1(K, \mu_{\hat{n}}))$  is the inclusion  $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/\hat{n}\mathbb{Z}$ . It is now immediate to check that the maps  $\hat{u}$  and  $u$  coincide.

We have thus obtained a map

$$(62) \quad \text{Ext}^1(\mathbb{Z}/n\mathbb{Z}, A_K) \rightarrow \text{Hom}(\pi_1(\mathbf{Gr}(A')), \mathbb{Q}/\mathbb{Z}), \quad \eta \mapsto u.$$

To check that this map is indeed a homomorphism, observe that it is functorial in  $A_K$ . Furthermore we could repeat the construction with any finite constant group  $F_K$  in place of  $\mathbb{Z}/n\mathbb{Z}$  obtaining in this way a map

$$\text{Ext}^1(F_K, A_K) \rightarrow \text{Hom}(\pi_1(\mathbf{Gr}(A')), \pi_0(\mathbf{H}^1(K, F_K^D)))$$

with  $F_K^D$  the Cartier dual of  $F_K$ . This construction is functorial in  $F_K$ . The functoriality results are sufficient to conclude that the map in (62) is a homomorphism, because the Baer sum of two extensions as in (40) is found by first taking the direct sum of the two extensions, then applying the push-out along the multiplication of  $A'_K$  and finally applying the pull-back along the diagonal map  $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ .

Suppose now that  $n$  and  $x_K$  are fixed. We show that the map  $u$  is trivial if  $X_K$  is trivial, *i.e.*, the map in (62) factors through  $H_{\mathbb{A}}^1(K, A_K) = \text{Ext}^1(\mathbb{Z}, A_K)$ . Suppose that  $X_K$  is trivial and choose a  $K$ -point  $x_K$  of  $X_K$ . In particular,  $V_{x_K}^* = \mathbb{G}_{m,K}$ ,  $T_K = \mathbb{G}_{m,K}$  and  $\pi_0(T) = \mathbb{Z}$ . Hence  $T^{\text{ft}} = \mathbb{G}_{m, \mathcal{O}_K}$ , the homomorphism  $u^{\text{ft}}: \pi_1(\text{Gr}(A')) \rightarrow \pi_0(\mathbf{Gr}(T^{\text{ft}})) = 0$  is the zero map and  $u = 0$ .

Suppose now that  $\text{char}(K) = 0$ . To see that the homomorphism  $X_K \mapsto u_{X_K}$  is Shafarevich's duality, it is sufficient to check that the homomorphisms  $u^\tau$  in (48) and  $u$  in (60) coincide. Consider then a finite separable extension  $K'/K$  splitting (41) and a point  $x_K = \text{Spec}(K')$  of  ${}_nE_K$  above 1. It is a rigidificator of  $\text{Pic}_{X_K/K}$ . Set  $y_K = {}_nE_K$ . Then, with the same notation as above,  $u^y$  coincides with the map  $u^\tau$  in (53) and one can repeat the arguments used in (61).  $\square$

**Remark 4.3.5.** The original construction by Bégueri works only for  $K$  of characteristic zero because in the case of positive characteristic the scheme  $V_{nE_K}^*$  (and hence  $T_K^\tau$ ) need not to be a torus; in particular it might not admit a Néron model. The construction via rigidificators described in this section works in any characteristic. For  $\text{char}(K) = p$  it is not clear that it provides Shafarevich's duality. We will see in Proposition 4.5.1 that this is the case on the prime-to- $p$  parts.

#### 4.4 A construction via the Picard functor

In this section we present a third possible construction of a homomorphism as in (7), this one making use of the relative Picard functor. We will see that it always coincides with the one in Proposition 4.3.4 and hence with Shafarevich's duality in the characteristic 0 case.

Let  $X_K$  be a torsor under  $A_K$  and  $x_K = \text{Spec}(K')$  a closed point of  $X_K$  with  $K'/K$  a finite separable extension. The smoothness of  $X_K$  ensures the existence of  $x_K$ . No assumption on the characteristic of  $K$  is made.

Consider the usual exact sequence (cf. [16], 2.4.1)

$$0 \rightarrow V_{X_K}^* \rightarrow V_{x_K}^* \rightarrow (\text{Pic}_{X_K/K}, x_K)^0 \rightarrow A'_K \rightarrow 0.$$

Observe that  $V_{X_K}^* = \mathfrak{R}_{X_K/K}(\mathbb{G}_{m, X_K}) = \mathbb{G}_{m, K}$  ([16], 2.4.3),  $V_{x_K}^*$  is a torus and hence so too is  $N_K := V_{x_K}^*/\mathbb{G}_{m, K}$ . Denote by  $\mathcal{N}$  its Néron model. Observe that it follows from Remark 4.3.2 that the component group of  $\mathcal{N}$  is cyclic of order  $n$ , hence its perfect Greenberg realization is a Serre pro-algebraic group.

We proceed as in the previous section, first by passing to Néron models and then applying the perfect Greenberg realization to the sequence

$$(63) \quad 0 \rightarrow N_K \rightarrow (\text{Pic}_{X_K/K}, x_K)^0 \xrightarrow{h_K} A'_K \rightarrow 0$$

so that we obtain an exact sequence of Serre pro-algebraic groups

$$0 \rightarrow \mathbf{Gr}(\mathcal{N}) \rightarrow \mathbf{Gr}(j_*(\text{Pic}_{X_K/K}, x_K)^0) \xrightarrow{h} \mathbf{Gr}(A') \rightarrow 0$$

and hence a homomorphism

$$(64) \quad v = v_{X_K}: \pi_1(\mathbf{Gr}(A')) \longrightarrow \pi_0(\mathbf{Gr}(\mathcal{N})) = \mathbb{Z}/n\mathbb{Z}.$$



In order to compare this construction with the (modified) Bégueri construction of the previous section, *i.e.*, in order to compare the maps  $u$  in (60) and  $v$  in (64), we consider the following diagram

$$(65) \quad \begin{array}{ccccccc} 0 & \longrightarrow & [V_{x_K}^*/\mu_n] = T_K & \longrightarrow & \underline{\text{Ext}}^1(E_K, \mathbb{G}_m)_{x_K} & \longrightarrow & A'_K \longrightarrow 0 \\ & & \downarrow t_K & & \downarrow f_K & & \parallel \\ 0 & \longrightarrow & [V_{x_K}^*/V_{X_K}^*] = N_K & \longrightarrow & (\text{Pic}_{X_K/K}, x_K)^0 & \xrightarrow{h_K} & A'_K \longrightarrow 0 \end{array}$$

where the upper sequence is (55), the lower one is (63) and  $f_K$  associates to a  $\mathbb{G}_m$ -extension  $L_K$  of  $E_K$  endowed with a  $x_K$ -section  $\sigma$  its restriction (as torsor) to  $X_K$  endowed with the trivialization along  $x_K$  induced by  $\sigma$ . The morphism  $t_K$  is surjective and its kernel is  $\mathbb{G}_{m,K} = V_{X_K}^*/\mu_n = \mathbb{G}_{m,K}/\mu_n$ .

Consider now the induced diagram on Néron models

$$\begin{array}{ccccccc} 0 & \longrightarrow & T^{\text{ft}} & \longrightarrow & G_1 & \longrightarrow & A'^0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & T^{\mathbb{C}} & \longrightarrow & j_* \underline{\text{Ext}}^1(E_K, \mathbb{G}_m)_{x_K} & \longrightarrow & A' \longrightarrow 0 \\ & & \downarrow & & \downarrow f & & \parallel \\ 0 & \longrightarrow & \mathcal{N} & \longrightarrow & j_*(\text{Pic}_{X_K/K}, x_K)^0 & \longrightarrow & A' \longrightarrow 0 \end{array}$$

$t^{\text{ft}}$  (curved arrow from  $T^{\text{ft}}$  to  $\mathcal{N}$ )

where the first row is (56). The homomorphism  $u$  in (60) is the composition of the homomorphism  $u^{\text{ft}}: \pi_1(\mathbf{Gr}(A'^0)) \rightarrow \pi_0(\mathbf{Gr}(T^{\text{ft}}))$  (deduced from the upper exact sequence) with the homomorphism

$$\pi_0(h^{\text{ft}}): \pi_0(T^{\text{ft}}) = \pi_0(\mathbf{Gr}(T^{\text{ft}})) \rightarrow \pi_0(\mathbf{H}^1(K, \mu_n)).$$

It now follows from the above diagram that the map  $v: \pi_1(\mathbf{Gr}(A')) \rightarrow \pi_0(\mathbf{Gr}(\mathcal{N}))$  in (64), obtained from the lower exact sequence, satisfies

$$(66) \quad v = \pi_0(t^{\text{ft}}) \circ u^{\text{ft}}.$$

We are going to check that  $u$  and  $v$  coincide, by showing that, up to canonical identifications we have  $\pi_0(h^{\text{ft}}) = \pi_0(t^{\text{ft}})$ . To see this fact, consider the following diagram

$$\begin{array}{ccccccc} & & & & \mathbb{G}_{m,K} & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & \mu_n & \longrightarrow & V_{x_K}^* & \longrightarrow & T_K \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow t_K \\ 0 & \longrightarrow & V_{X_K}^* = \mathbb{G}_{m,K} & \longrightarrow & V_{x_K}^* & \longrightarrow & N_K \longrightarrow 0 \\ & & \downarrow n & & & & \\ & & \mathbb{G}_{m,K} & & & & \end{array}$$

and consider the induced diagram of component groups of Néron models

$$\begin{array}{ccccccc}
& & & & \mathbb{Z} & & \pi_0(T^{\text{ft}}) = \pi_0(T)_{\text{tor}} \\
& & & & \downarrow & & \swarrow \iota \\
& & \pi_0(V_{x_K}^*) & \longrightarrow & \pi_0(T) & & \\
& & \parallel & & \downarrow & & \swarrow \pi_0(t^{\text{ft}}) \\
0 \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_0(V_{x_K}^*) & \longrightarrow & \pi_0(\mathcal{N}) & \longrightarrow 0
\end{array}$$

where  $\iota$  is the inclusion map and the vertical sequence is left exact because  $\mathbb{Z}$  is torsion free (cf. [8], VIII 5.5). We insert this diagram into a bigger diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathbb{Z} & \xrightarrow{n} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \parallel & \\
0 \longrightarrow & \pi_0(V_{x_K}^*) & \longrightarrow & \pi_0(T) & \xrightarrow{q_1} & \pi_0(\mathbf{H}^1(K, \mu_n)) & \longrightarrow 0 \\
& \downarrow & & \downarrow q_2 & & \uparrow \pi_0(h^{\text{ft}}) & \\
& \pi_0(\mathcal{N}) & \xlongequal{\quad} & \pi_0(\mathcal{N}) & \xleftarrow{\pi_0(t^{\text{ft}})} & \pi_0(T^{\text{ft}}) &
\end{array}$$

where  $q_1 \circ \iota = \pi_0(h^{\text{ft}})$  and  $q_2 \circ \iota = \pi_0(t^{\text{ft}})$ . By Remark 4.3.2,  $\pi_0(\mathcal{N}) = \mathbb{Z}/n\mathbb{Z}$  and the vertical sequence on the left coincides with the upper horizontal sequence. Hence the middle vertical sequence splits as does the middle horizontal sequence. The identifications  $\pi_0(V_{x_K}^*) = \mathbb{Z}$  then induce the identifications

$$\pi_0(\mathcal{N}) \cong \mathbb{Z}/n\mathbb{Z} \cong \pi_0(\mathbf{H}^1(K, \mu_n))$$

where the first isomorphism maps the image of the class of a uniformizer  $\pi' \in K'^* = V_{x_K}^*(K)$  to the class of 1, while the second isomorphism maps the class of 1 to the image of the cohomology class corresponding to a uniformizer  $\pi \in K^* = \mathbb{G}_{m,K}(K)$ .

Let  $\sigma$  be a section of  $q_2$ . One has  $q_1 \circ \sigma = \text{id}_{\mathbb{Z}/n\mathbb{Z}}$ . Furthermore  $\sigma \circ q_2 \circ \iota = \iota$  because  $\sigma \circ q_2 \circ \iota - \iota$  factors through  $\mathbb{Z}$  and thus is trivial because  $\pi_0(T^{\text{ft}})$  is torsion. Hence

$$\pi_0(h^{\text{ft}}) = q_1 \circ \iota = q_1 \circ \sigma \circ q_2 \circ \iota = q_2 \circ \iota = \pi_0(t^{\text{ft}})$$

and hence thanks to (66) and (60), we get

$$v = \pi_0(t^{\text{ft}}) \circ u^{\text{ft}} = \pi_0(h^{\text{ft}}) \circ u^{\text{ft}} = u.$$

We can then state the main result which is an immediate consequence of what we have just proved and Proposition 4.3.4:

**Theorem 4.4.1.** *Let  $A_K$  be an abelian variety over  $K$ . The homomorphism*

$$\Xi: \mathbf{H}^1(K, A_K) \rightarrow \text{Hom}(\pi_1(\mathbf{Gr}(A')), \mathbb{Q}/\mathbb{Z})$$

*mapping the torsor  $X_K$  to the homomorphism  $u_{X_K}: \pi_1(\mathbf{Gr}(A')) \rightarrow \mathbb{Q}/\mathbb{Z}$  in (60) coincides with the homomorphism mapping  $X_K$  to the homomorphism  $v_{X_K}$  in (64). If furthermore the characteristic of  $K$  is zero, then both constructions coincide with B  queri's construction in (48), i.e., they explicate Shafarevich's duality.*

## 4.5 Comparison between the constructions in § 4.3.1 and § 4.4 for general $K$

We have seen in Theorem 4.4.1 that the homomorphisms in (60), (64) always coincide and in the mixed characteristic case, that they are isomorphisms and that they provide Shafarevich's duality. For  $K$  of characteristic  $p$ , it is not clear in general either that they are isomorphisms or that they provide Shafarevich's duality. However, we have a partial result on the prime-to- $p$  parts where Shafarevich's duality is quite easy to describe.

### 4.5.1 Shafarevich's duality on the prime-to- $p$ parts

We recall here what Shafarevich's duality looks like on the prime-to- $p$  parts.

Let  $n = l^r$  be a positive integer, prime to  $p$ , and large enough to kill the  $l$ -primary parts of the component groups of  $A_K$  and  $A'_K$ . Consider the perfect cup product pairing

$$\langle \cdot, \cdot \rangle: H^1(K, {}_n A_K) \times {}_n A'_K(K) \rightarrow H^1(K, \mu_n) = \mathbb{Z}/n\mathbb{Z}$$

on the (étale or flat) cohomology groups of the  $n$ -torsion points of  $A_K$  and  $A'_K$ . Given an extension  $\eta_n$  as in (41) (which corresponds to the torsor  $X_K$ ) and a point  $a \in {}_n A'_K(K)$ , then  $\langle \eta_n, a \rangle$  is the class of the pull-back along  $a: \mathbb{Z} \rightarrow {}_n A'_K$  of the Cartier dual of  $\eta_n$ ,

$$\eta_n^D: 0 \rightarrow \mu_n \rightarrow {}_n E_K^D \rightarrow {}_n A'_K \rightarrow 0,$$

and it corresponds to the image of  $a$  along the boundary map  $\partial: {}_n A'_K(K) \rightarrow H^1(K, \mu_n)$ . Furthermore, if  ${}_n A^0$  denotes the quasi-finite subgroup of  $n$ -torsion sections of  $A^0$ , we have

$$\pi_1(\mathbf{Gr}(A'))/n\pi_1(\mathbf{Gr}(A')) = {}_n A^0(\mathcal{O}_K) = {}_n A'(\mathcal{O}_K)/{}_n A'(\mathcal{O}_K),$$

$${}_n H^1(K, A_K) = H^1(K, {}_n A_K)/H^1(K, {}_n A_K)$$

(cf. [2] § 1) and Shafarevich's duality on the  $n$ -primary parts

$${}_n H^1(K, A_K) \times \pi_1(\mathbf{Gr}(A'))/n\pi_1(\mathbf{Gr}(A')) \rightarrow H^1(K, \mu_n) = \mathbb{Z}/n\mathbb{Z}$$

is induced by the above cup product.

The map  $u^\tau: \pi_1(\mathbf{Gr}(A')) \rightarrow \pi_0(H^1(K, \mu_n)) = H^1(K, \mu_n)$  in (48) associated with the torsor  $X_K$  can also be viewed as the composition

$$(67) \quad \pi_1(\mathbf{Gr}(A')) \xrightarrow{\delta} {}_n A'(\mathcal{O}_K) = {}_n A'_K(K) \xrightarrow{\partial} H^1(K, \mu_n) = \mathbb{Z}/n\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$$

where the first map is deduced from the exact sequence

$$0 \rightarrow {}_n A'_K \rightarrow A'_K \xrightarrow{n} A'_K \rightarrow 0$$

on passing to Néron models. More precisely we have

$$0 \rightarrow {}_n A' \rightarrow A' \xrightarrow{n} {}_n A' \rightarrow 0$$

where  ${}_n A'$  is a subgroup scheme of  $A'$  that contains  $A^0$ . In particular, on applying the perfect Greenberg realization functor we get a homomorphism

$$(68) \quad \pi_1(\mathbf{Gr}(A')) = \pi_1(\mathbf{Gr}({}_n A')) \rightarrow \pi_0({}_n A') = {}_n A'(\mathcal{O}_K).$$

#### 4.5.2 Comparison results on the prime-to- $p$ parts

Let  $X_K$  be a torsor under  $A_K$  of order  $d$  with  $d$  a power of a prime integer  $l$ ,  $l \neq p$ . Let  $n = l^r$  be a multiple of  $d$  large enough to kill the  $l$ -primary parts of the component groups of  $A_K$  and  $A'_K$ . Fix an extension corresponding to  $X_K$  as in (40), and let  $x_K = \text{Spec}(K')$  be a rigidificator of  $\text{Pic}_{X_K/K}$  contained in  ${}_nE_K$ , i.e., a point of  ${}_nE_K$  above  $1 \in \mathbb{Z}/n\mathbb{Z}$  in (41). We show that the composition of the maps in (67) coincides with the map  $u$  in (60). This is sufficient to conclude that our construction via rigidificators (or equivalently via the relative Picard functor) is Shafarevich's duality on the prime-to- $p$  parts.

With notations as in (40), observe that the  $n$ -multiplication on  $A_K$  factors through  $E_K$  so that we have a homomorphism  $\gamma: E_K \rightarrow A_K$ , with kernel  ${}_nE_K$  such that  $\gamma \circ \alpha = n$ . Consider the sequence in (54). We have a diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mu_n & \longrightarrow & {}_nE_K^D & \longrightarrow & \underline{\text{Ext}}^1(A_K, \mathbb{G}_m) & \xrightarrow{n} & A'_K \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \gamma^* & & \parallel \\ 0 & \longrightarrow & \mu_n = \underline{\text{Hom}}(E_K, \mathbb{G}_m) & \longrightarrow & V_{x_K}^* & \longrightarrow & \underline{\text{Ext}}^1(E_K, \mathbb{G}_m)_{x_K} & \longrightarrow & A'_K \longrightarrow 0. \end{array}$$

Indeed  ${}_nE_K^D = \underline{\text{Hom}}({}_nE_K, \mathbb{G}_m)$  maps canonically to  $V_{x_K}^* = \underline{\text{Mor}}(x_K, \mathbb{G}_{m,K})$ ; hence  ${}_nA'_K$  maps to the torus  $T_K = V_{x_K}^*/\mu_n$  in (55). The push-out of the exact sequence  $0 \rightarrow {}_nA'_K \rightarrow A'_K \rightarrow A'_K \rightarrow 0$  along  ${}_nA'_K \rightarrow T_K$  provides the sequence (55) and the homomorphism  $\gamma^*$  sends a  $\mathbb{G}_m$ -extension of  $A_K$  to its pull-back along  $\gamma$  endowed with its canonical trivialization along  $x_K$ , induced by the canonical trivialization along  ${}_nE_K$ . Moreover, the boundary map  $\partial: {}_nA'_K(K) \rightarrow H^1(K, \mu_n)$  (of finite groups) is the composition of  $\nu: {}_nA'_K(K) \rightarrow T_K(K)$  with the boundary map  $h: T_K(K) \rightarrow H^1(K, \mu_n)$ , i.e.,

$$(69) \quad \partial = h \circ \nu.$$

Recall furthermore that the kernel of the  $n$ -multiplication on  $A'$  is a quasi-finite group scheme over  $\mathcal{O}_K$  whose finite part is an étale finite group scheme over  $\mathcal{O}_K$  of order prime to  $p$ , hence constant, because  $\mathcal{O}_K$  is strictly henselian. On the level of pro-algebraic groups we then have a diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}_nA'(\mathcal{O}_K) & \longrightarrow & \mathbf{Gr}(A') & \longrightarrow & \mathbf{Gr}({}_nA') & \longrightarrow & 0 \\ & & \downarrow \nu & & \downarrow \alpha^* & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Gr}(T) & \longrightarrow & \mathbf{Gr}(j_* \underline{\text{Ext}}^1(E_K, \mathbb{G}_m)_{x_K}) & \longrightarrow & \mathbf{Gr}(A') & \longrightarrow & 0 \end{array}$$

Since the vertical map on the left factors through a map  $\nu^{\text{ft}}: {}_nA'(\mathcal{O}_K) \rightarrow \mathbf{Gr}(T^{\text{ft}})$ , the homomorphism  $u^{\text{ft}}: \pi_1(\mathbf{Gr}(A')) \rightarrow \pi_0(\mathbf{Gr}(T^{\text{ft}})) = \pi_0(\mathbf{Gr}(T))_{\text{tor}}$  in (60) factors through the map  $\delta: \pi_1(\mathbf{Gr}(A')) \rightarrow {}_nA'(\mathcal{O}_K)$  in (68) and hence

$$u^\tau \stackrel{(67)}{=} \partial \circ \delta \stackrel{(69)}{=} h \circ \nu \circ \delta = \pi_0(h^{\text{ft}}) \circ \pi_0(\nu^{\text{ft}}) \circ \delta = \pi_0(h^{\text{ft}}) \circ u^{\text{ft}} = u,$$

i.e., the homomorphism  $u: \pi_1(\mathbf{Gr}(A')) \rightarrow \mathbb{Z}/n\mathbb{Z}$  in (60) coincides with that in (67). Thus we have

**Proposition 4.5.1.** *For any local field  $K$  with algebraically closed residue field, Shafarevich's pairing induces the homomorphism  $\Xi$  in Proposition 4.3.4, on the prime-to- $p$  parts.*

The comparison for the  $p$  parts in the equal positive characteristic case is still open.

## 5 Comparison between (5) and (7)

In this last section, we return to the study of torsors under an elliptic curve  $A_K$ , and we examine the relation between the fundamental short exact sequence (2) of Serre pro-algebraic groups with Shafarevich's duality of abelian varieties. Let  $n \geq 1$  be an integer, and  $X_K$  a torsor under the elliptic curve  $A_K$  of order  $d$  dividing  $n$ . Thanks to Corollary 3.4.4 and the construction in §3.1 we are provided with a short exact sequence of Serre pro-algebraic groups

$$(70) \quad 0 \rightarrow \mathbb{Z}/d\mathbb{Z} \rightarrow \mathbf{Pic}^0(\mathbf{X}) \xrightarrow{q} \mathbf{J}(\mathbf{S}) \rightarrow 0$$

by sending  $\bar{1} \in \mathbb{Z}/d\mathbb{Z}$  to  $\mathcal{O}_X(-D) \in \mathbf{Pic}^0(\mathbf{X})$ . If we push out this short exact sequence by the canonical map

$$(71) \quad \mathbb{Z}/d\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}, \quad \bar{1} \mapsto \frac{d}{n} \cdot \bar{1},$$

we get an element, denoted by  $\Phi_n(X_K)$ , of the group  $\mathrm{Ext}_k^1(\mathbf{Gr}(J), \mathbb{Z}/n\mathbb{Z})$  of extensions of the pro-algebraic group  $\mathbf{Gr}(J) = \mathbf{J}(\mathbf{S})$  by the constant group  $\mathbb{Z}/n\mathbb{Z}$ . In this way we get the following canonical map of sets:

$$(72) \quad \Phi_n: {}_n\mathrm{H}_{\mathbb{A}}^1(K, A_K) \rightarrow \mathrm{Ext}_k^1(\mathbf{Gr}(J), \mathbb{Z}/n\mathbb{Z}).$$

Motivated by [19], one might ask if this morphism is always an isomorphism. Our strategy in studying this question is to relate the above construction to Shafarevich's pairing in (7) by using our new construction in § 4 as an intermediate bridge.

### 5.1 Some set theoretical considerations

As usual, we use  $X_K$  to denote a torsor under  $A_K$  of order  $d$  and  $X$  for its (proper)  $S$ -minimal regular model. We begin with the following lemma, which follows from Lemma 2.0.2.

**Lemma 5.1.1.** *The schematic closure  $Y$  in  $X$  of any closed point  $x_K$  of  $X_K$  provides a rigidification of the Picard functor  $\mathrm{Pic}_{X/S}$ .*

*Proof.* We need only verify the injectivity of the map  $\mathrm{H}^0(X_s, \mathcal{O}_{X_s}) \rightarrow \mathrm{H}^0(Y_s, \mathcal{O}_{Y_s})$  (Corollary 2.2.2 of [16]). More generally, we will prove by induction on  $n$  that the canonical morphism  $\mathrm{H}^0(X_n, \mathcal{O}_{X_n}) \rightarrow \mathrm{H}^0(Y_n, \mathcal{O}_{Y_n})$  is injective, where  $Y_n := Y \times_X X_n$ . Let  $Y$  be defined by the ideal sheaf  $\mathcal{J}$ . Then  $Y_n$  is defined by the ideal sheaf  $\mathcal{I}^n + \mathcal{J}$ . Let us begin with the case  $n = 1$ : by Lemma 2.0.2, we know that  $\mathrm{H}^0(X_1, \mathcal{O}_{X_1}) = k$ . Let  $\varepsilon \in \mathrm{H}^0(X_1, \mathcal{O}_{X_1})$ ; then  $\varepsilon$  is a global function on  $X_1$  and so is constant. As a result, the image of  $\varepsilon$  in  $\mathrm{H}^0(Y_1, \mathcal{O}_{Y_1})$  is zero if and only if  $\varepsilon = 0$  that is, the morphism  $\mathrm{H}^0(X_1, \mathcal{O}_{X_1}) \rightarrow \mathrm{H}^0(Y_1, \mathcal{O}_{Y_1})$  is injective. In order to complete the induction, consider the following diagram of sheaves over  $X$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \frac{\mathcal{I}^n}{\mathcal{I}^{n+1}} & \longrightarrow & \frac{\mathcal{O}_X}{\mathcal{I}^{n+1}} & \longrightarrow & \frac{\mathcal{O}_X}{\mathcal{I}^n} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \frac{\mathcal{I}^n + \mathcal{J}}{\mathcal{I}^{n+1} + \mathcal{J}} & \longrightarrow & \frac{\mathcal{O}_X}{\mathcal{I}^{n+1} + \mathcal{J}} & \longrightarrow & \frac{\mathcal{O}_X}{\mathcal{I}^n + \mathcal{J}} \longrightarrow 0 \end{array}$$

where  $\mathrm{H}^0(X, \frac{\mathcal{O}_X}{\mathcal{I}^m}) = \mathrm{H}^0(X_m, \mathcal{O}_{X_m})$  and  $\mathrm{H}^0(X, \frac{\mathcal{O}_X}{\mathcal{I}^m + \mathcal{J}}) = \mathrm{H}^0(Y_m, \mathcal{O}_{Y_m})$ . Hence we need only establish the injectivity of the morphism

$$\mathrm{H}^0\left(X, \frac{\mathcal{I}^n}{\mathcal{I}^{n+1}}\right) \rightarrow \mathrm{H}^0\left(X, \frac{\mathcal{I}^n + \mathcal{J}}{\mathcal{I}^{n+1} + \mathcal{J}}\right).$$

Observe first that  $\frac{\mathcal{I}^n}{\mathcal{I}^{n+1}} \simeq \mathcal{I}^n \otimes_{\mathcal{O}_X} \frac{\mathcal{O}_X}{\mathcal{I}}$ . Hence

$$H^0\left(X, \frac{\mathcal{I}^n}{\mathcal{I}^{n+1}}\right) \simeq H^0\left(X, \mathcal{I}^n \otimes_{\mathcal{O}_X} \frac{\mathcal{O}_X}{\mathcal{I}}\right) = H^0(X_1, \mathcal{I}^n|_{X_1}).$$

Furthermore, consider the map  $\frac{\mathcal{I}^n}{\mathcal{I}^{n+1}} \otimes_{\mathcal{O}_X} \frac{\mathcal{O}_X}{\mathcal{I} + \mathcal{J}} \rightarrow \frac{\mathcal{I}^n + \mathcal{J}}{\mathcal{I}^{n+1} + \mathcal{J}}$  that, on sections, maps  $\bar{a} \otimes \bar{b}$  to  $\overline{ab}$ . It is well defined and surjective. Since  $\mathcal{I}$  is invertible,  $Y$  is integral and  $Y \not\subseteq X_1$ , so our map is also injective. In particular, we find

$$\frac{\mathcal{I}^n + \mathcal{J}}{\mathcal{I}^{n+1} + \mathcal{J}} \simeq \frac{\mathcal{I}^n}{\mathcal{I}^{n+1}} \otimes_{\mathcal{O}_X} \frac{\mathcal{O}_X}{\mathcal{I} + \mathcal{J}} \simeq \mathcal{I}^n \otimes_{\mathcal{O}_X} \frac{\mathcal{O}_X}{\mathcal{I}} \otimes_{\mathcal{O}_X} \frac{\mathcal{O}_X}{\mathcal{I} + \mathcal{J}} \simeq \mathcal{I}^n \otimes_{\mathcal{O}_X} \frac{\mathcal{O}_X}{\mathcal{I} + \mathcal{J}}.$$

Hence

$$H^0\left(X, \frac{\mathcal{I}^n + \mathcal{J}}{\mathcal{I}^{n+1} + \mathcal{J}}\right) \simeq H^0\left(X, \mathcal{I}^n \otimes_{\mathcal{O}_X} \frac{\mathcal{O}_X}{\mathcal{I} + \mathcal{J}}\right) = H^0(Y_1, \mathcal{I}^n|_{Y_1}).$$

We are then reduced to proving that the restriction map

$$H^0(X_1, \mathcal{I}^n|_{X_1}) \longrightarrow H^0(Y_1, \mathcal{I}^n|_{Y_1})$$

is injective. Since  $\mathcal{I}$  is an invertible sheaf, according to Lemma 2.0.2, the first group is trivial or its consists of constant functions; hence the result follows.  $\square$

Let  $x_K = \text{Spec}(K')$  be a closed point of  $X_K$  with  $K'/K$  a finite *separable* extension of degree  $d$  (this is possible thanks to [6], 8.4 (3)), and take  $Y = \overline{\{x_K\}} \subset X$  the schematic closure of  $x_K$ . By Lemma 5.1.1, the subscheme  $Y \hookrightarrow X$  is a rigidificator for the Picard functor  $\text{Pic}_{X/S}$  of the curve  $X/S$ . We can then consider the rigidified Picard functor of  $X/S$  along the subscheme  $Y$  as recalled in § 1.1. Moreover, as we have seen in the proof of Lemma 2.1.2, such a divisor  $Y$  is necessarily integral and *regular*, and it cuts a unique component  $C_i$  of multiplicity 1 in  $D$  transversally. In the following, we will use notations as in § 1.1 and § 3. In particular,  $G = (\text{Pic}_{X/S}, Y)^0$  is the identity component of the rigidified Picard scheme  $(\text{Pic}_{X/S}, Y)$ , and we have the following canonical map

$$r: G = (\text{Pic}_{X/S}, Y)^0 \rightarrow \text{Pic}_{X/S}^0$$

that forgets rigidifications. Let  $N$  be the kernel of the morphism  $r$ . In general, this fppf-sheaf  $N$  is not representable, but it has representable fibres. Following § 3, we denote by  $H = \overline{N_K} \hookrightarrow (\text{Pic}_{X/S}, Y)^0 = G$  the schematic closure of  $N_K$  in  $G$ ; it is representable by a flat  $S$ -group scheme of finite type. Then the fppf quotient  $J = G/H$  gives us the identity component of the  $S$ -Néron model of the Jacobian  $J_K = \text{Pic}_{X_K/K}^0$  of the curve  $X_K/K$ , and one has the following exact sequence of  $S$ -group schemes:

$$0 \longrightarrow H \longrightarrow G \xrightarrow{\theta} J \longrightarrow 0.$$

which induces an exact sequence of abstract groups (§ 1.1.2):

$$(73) \quad 0 \longrightarrow H(S) \longrightarrow G(S) \longrightarrow J(S) \longrightarrow 0.$$

On the other hand, by definition, we have another exact sequence of sheaves, which is exact for the étale topology (since (8) in § 1.1 is exact for the étale topology):

$$0 \longrightarrow N \longrightarrow G \xrightarrow{r} \text{Pic}_{X/S}^0 \longrightarrow 0.$$



Since  $S$  is strictly henselian, the latter sequence induces the following short exact sequence of abstract groups:

$$(74) \quad 0 \longrightarrow N(S) \longrightarrow G(S) \longrightarrow \mathrm{Pic}^0(X) \longrightarrow 0.$$

On combining (73) and (74), we get the following commutative diagram of abstract groups with exact rows:

$$(75) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & N(S) & \longrightarrow & G(S) & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & 0 \\ & & \downarrow & & \parallel & & \downarrow & & \\ 0 & \longrightarrow & H(S) & \longrightarrow & G(S) & \longrightarrow & J(S) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N(K) & \longrightarrow & G(K) & \longrightarrow & \mathrm{Pic}_{X/S}^0(K) & \longrightarrow & 0 \end{array}$$

where the lower sequence is exact on the right because  $N_K$  is a torus, the upper vertical map on the right is surjective ([5], 9.5/2) and the remaining vertical maps are all injective.

## 5.2 The pro-algebraic nature of diagram (75)

For  $n \in \mathbb{Z}_{\geq 1}$ , as usual we put  $S_n = \mathrm{Spec}(\mathcal{O}_{K,n}) = \mathrm{Spec}(\mathcal{O}_K/\pi^n)$  with  $\pi \in \mathcal{O}_K$  a uniformizer and we denote by  $\mathbb{R}_n$  the Greenberg algebra associated with  $\mathcal{O}_{K,n}$  (Appendix A of [10]). The aim of this subsection is to show, with the help of Greenberg realization functors, that the diagram (75) is pro-algebraic in nature.

First, the sheaf  $H$  is representable by an  $S$ -group scheme separated of finite type. Hence its Greenberg realization  $\mathrm{Gr}_n(H)$  is representable by a  $k$ -scheme of finite type (§ 1.2) and we have the following short exact sequence:

$$0 \rightarrow \mathrm{Gr}_n(H) \rightarrow \mathrm{Gr}_n(G) \rightarrow \mathrm{Gr}_n(J) \rightarrow 0.$$

For the right exactness, we need only prove that the map  $\mathrm{Gr}_n(G) \rightarrow \mathrm{Gr}_n(J)$  induces a surjective map on the groups of  $k$ -rational points, *i.e.*, that the morphism of group  $G(S_n) \rightarrow J(S_n)$  is surjective. This last statement follows from the surjectivity of the maps  $\theta(S): G(S) \rightarrow J(S)$  (see § 1.1.2) and  $J(S) \rightarrow J(S_n)$ . On passing to the projective limit of the associated perfect group schemes, one obtains an extension of Serre pro-algebraic groups

$$0 \rightarrow \mathbf{Gr}(H) \rightarrow \mathbf{Gr}(G) \rightarrow \mathbf{Gr}(J) \rightarrow 0$$

which says that (73) is pro-algebraic in nature.

Next, we consider the fppf sheaf  $N$  kernel of  $r: G \rightarrow \mathrm{Pic}_{X/S}$ . Let us first remark that for any  $k$ -algebra  $A$ , by considering the  $\mathcal{O}_K$ -algebra  $\mathbb{R}_n(A)$  (with support over  $S_n \hookrightarrow S$ ), we have the following exact sequence of groups:

$$0 \rightarrow N(\mathbb{R}_n(A)) \rightarrow G(\mathbb{R}_n(A)) \rightarrow \mathrm{Pic}_{X/S}^0(\mathbb{R}_n(A)).$$

Let  $\mathrm{Gr}_n(N)$  be the fppf sheaf associated with the pre-sheaf  $A \mapsto N(\mathbb{R}_n(A))$ . By taking the associated fppf sheaves, we get the following exact complex of algebraic  $k$ -groups (where the representability of  $\mathrm{Gr}_n(N)$  follows from the representability of the last two functors by smooth group schemes):

$$(76) \quad 0 \rightarrow \mathrm{Gr}_n(N) \rightarrow \mathrm{Gr}_n(G) \rightarrow \mathrm{Gr}_n(\mathrm{Pic}_{X/S}^0).$$

By taking the  $k$ -rational points, we get the usual exact sequence

$$0 \rightarrow N(S_n) \rightarrow G(S_n) \rightarrow \mathrm{Pic}_{X/S}^0(S_n) \rightarrow 0$$

which is exact on the right since  $\mathcal{O}_{K,n}$  is strictly henselian. So the complex (76) is in fact a short exact sequence of algebraic  $k$ -groups. Now, by taking the projective limit with respect to  $n$  in the sequence of perfect groups associated with (76) we get a short exact sequence of Serre pro-algebraic groups:

$$0 \rightarrow \mathbf{N}(\mathbf{S}) \rightarrow \mathbf{Gr}(G) \rightarrow \mathbf{Pic}^0(\mathbf{X}) \rightarrow 0.$$

Finally, the group scheme  $N_K$  is a torus,  $\mathrm{Pic}_{X_K/K}^0 = A'_K$  is an elliptic curve and  $G_K = (\mathrm{Pic}_{X/S}, Y)_K^0$  is a semi-abelian variety; hence they all admit Néron models, which will be denoted by  $\mathcal{N}$ ,  $A'$  and  $\mathcal{G}$  respectively; in particular they are smooth group schemes over  $S$ . Moreover, according to Remark 4.3.2, the  $S$ -group schemes  $\mathcal{N}$ ,  $A'$  are of finite type over  $S$ , and hence the same holds for  $\mathcal{G}$ .

By the Néron mapping property we have the following two canonical maps

$$f_G: G \rightarrow \mathcal{G}, \quad \text{and} \quad f_A: J \rightarrow A'.$$

As a consequence, the morphisms

$$G(S) \rightarrow G(K) = \mathcal{G}(S), \quad \text{and} \quad J(S) \rightarrow \mathrm{Pic}_{X/S}^0(K) = A'(S)$$

in diagram (75) come from the morphisms of Serre pro-algebraic groups

$$\mathbf{Gr}(G) \rightarrow \mathbf{Gr}(\mathcal{G}), \quad \text{and} \quad \mathbf{Gr}(J) \rightarrow \mathbf{Gr}(A')$$

induced by  $f_G, f_A$ . This implies the existence of a morphism of pro-algebraic groups

$$H(S) \rightarrow N(K) = \mathbf{Gr}(\mathcal{N})$$

which realizes the lower left vertical inclusion in (75). Summarizing, we have that (75) comes from a commutative diagram (with exact rows) of Serre pro-algebraic groups.

$$(77) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{N}(\mathbf{S}) & \longrightarrow & \mathbf{Gr}(G) & \longrightarrow & \mathbf{Pic}^0(\mathbf{X}) \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & \mathbf{H}(\mathbf{S}) & \longrightarrow & \mathbf{Gr}(G) & \longrightarrow & \mathbf{Gr}(J) \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbf{N}(\mathbf{K}) & \longrightarrow & \mathbf{Gr}(\mathcal{G}) & \longrightarrow & \mathbf{Pic}_{X/S}^0(\mathbf{K}) = \mathbf{Gr}(A') \longrightarrow 0 \end{array}$$

### 5.3 Comparison

We deduce from (77) a commutative diagram of profinite groups:

$$(78) \quad \begin{array}{ccc} \pi_1(\mathbf{Gr}(J)) & \xrightarrow{\simeq} & \pi_1(\mathbf{Gr}(A')) \\ \downarrow & & \downarrow \\ \pi_0(\mathbf{H}(\mathbf{S})) & \xrightarrow{\alpha} & \pi_0(\mathbf{N}(\mathbf{K})) \end{array}$$

The upper arrow is an isomorphism because  $J = A^0$ , hence  $\mathbf{Gr}(J) = \mathbf{Gr}(A')^0$ .

In order to give an explicit description of the morphism  $\alpha$ , recall first of all that the group of connected components  $\pi_0(\mathbf{N}(\mathbf{K})) = \pi_0(\mathcal{N})$  is isomorphic to  $\mathbb{Z}/d\mathbb{Z}$  (cf. Remark 4.3.2 and (65)), with identification given by

$$\beta: \pi_0(\mathbf{N}(\mathbf{K})) \rightarrow \mathbb{Z}/d\mathbb{Z}, \quad \text{class of } \pi' \text{ in } N_K(K) = k(x_K)^*/K^* \mapsto \bar{1}$$

where  $\pi' \in k(x_K)$  a uniformizer. Furthermore, the class of  $\pi'$  in  $N_K(K)$ , viewed as element of  $G(K) = (\text{Pic}_{X/S}, Y)^0(K)$  (see § 1.1), is the trivial line bundle on  $X_K$  with the rigidification on  $Y_K$  given by the multiplication by  $\pi'$ .

Second, the component group of  $\mathbf{H}(\mathbf{S})$  is also  $\mathbb{Z}/d\mathbb{Z}$ . Indeed our group scheme  $H$  coincides with the one denoted by  $H_1$  in [11], pp. 18–21. We then have the following exact sequence

$$V_Y^*(S) \rightarrow H(S) \rightarrow \mathbb{Z}/d\mathbb{Z} \rightarrow 0,$$

(*loc. cit.*, Theorem 3.5) where the first map is the natural factorization of  $V_Y^* \rightarrow N \hookrightarrow G := (\text{Pic}_{X/S}, Y)^0$  through  $H \hookrightarrow G$  (since  $V_Y$  is flat over  $S$ ), and the second map is defined by

$$\gamma: H(S) \rightarrow \mathbb{Z}/d\mathbb{Z}, \quad \left( \mathcal{O}_X \left( \frac{m}{d} X_s \right), a \right) \mapsto \bar{m} \in \mathbb{Z}/d\mathbb{Z}.$$

(see [11], 3.5). Applying the perfect Greenberg functor to the morphisms  $V_X^* \rightarrow V_Y^* \rightarrow G$  one sees that the map  $\gamma$  is of pro-algebraic nature and we write:

$$\gamma: \pi_0(\mathbf{H}(\mathbf{S})) \xrightarrow{\sim} \mathbb{Z}/d\mathbb{Z}$$

**Lemma 5.3.1.** *Maintain the notation used above. The following diagram is commutative*

$$(79) \quad \begin{array}{ccc} \pi_0(\mathbf{H}(\mathbf{S})) & \xrightarrow{\alpha} & \pi_0(\mathbf{N}(\mathbf{K})) \\ \downarrow \gamma & & \downarrow \beta \\ \mathbb{Z}/d\mathbb{Z} & \xrightarrow{\bar{1} \mapsto -\bar{1}} & \mathbb{Z}/d\mathbb{Z} \end{array}$$

*In particular, the morphism  $\alpha$  is an isomorphism.*

*Proof.* Recall that we denote by  $D$  the vertical divisor  $\frac{1}{d}X_s$ . Since the map  $H(S) \rightarrow N(K)$  sends  $(\mathcal{O}_X(-D), a)$  to its generic fibre, and the generic fibre of  $\mathcal{O}_X(-D)$  is trivial, we are reduced to verifying that a rigidification on  $Y_K$  can be given by multiplication by the uniformizer  $\pi'$ , and that this rigidification extends to a rigidification of  $\mathcal{O}_X(-D)$  on  $Y$ .

Now consider  $\mathcal{O}_X(-D)$ . This gives us an ideal sheaf of  $\mathcal{O}_X$ . Recall that  $Y$  is regular, hence  $Y = \text{Spec}(R')$  with  $R'$  a complete discrete valuation ring whose field of fractions is  $K' := k(x_K)$ . Next, we claim that the intersection of  $Y$  and  $D$ , viewed as a divisor of  $Y$ , is defined by the equation  $\pi' = 0$ . In fact, let  $y \in Y$  be its closed point, and consider the local ring  $\mathcal{O}_{X,y}$  which is regular of dimension 2. Let  $r \in \mathcal{O}_{X,y}$  (respectively  $t \in \mathcal{O}_{X,y}$ ) be a defining equation of  $Y$  (respectively of  $D$ ) around  $y \in X$ . Since  $X_s = dD$  as divisor of  $X$ , we have  $(\pi) = (t^d) \subset \mathcal{O}_{X,y}$ . By definition,  $Y/S$  is of degree  $d$ ; it follows that the intersection number in  $X$

$$Y \cdot X_k = \ell(\mathcal{O}_{X,y}/(r, \pi)) = \ell(\mathcal{O}_{X,y}/(r, t^d)) = \ell(R'/(t')^d)$$

is equal to  $d$ , with  $t'$  the image of  $t$  in  $R' = \mathcal{O}_{X,y}/(r)$ . This implies that  $t'$  is an uniformizer of  $R'$  (and the maximal ideal of  $\mathcal{O}_{X,y}$  is generated by  $r$  and  $t$ ). Hence  $t' = u'\pi'$  with  $u' \in R'$  a

unit. As a result, the intersection  $Y \cap D$  is defined by the equation  $t' = 0$ , or equivalently, by the equation  $\pi' = 0$  in  $Y$ .

So by the claim, we have  $\mathcal{O}_Y(-D \cap Y) = (\pi')$ . We get in this way a rigidification of  $\mathcal{O}_X(-D)$  along  $Y$

$$a: \widetilde{R}' = \mathcal{O}_Y \rightarrow \mathcal{O}_X(-D)|_Y = (\pi'), \quad 1 \mapsto \pi',$$

with  $\widetilde{R}'$  the coherent module associated with  $R'$ . Now, if we restrict to the generic point, we get

$$a_K: \widetilde{K}' = \mathcal{O}_{Y_K} \rightarrow \mathcal{O}_X(-D)|_{Y_K} = (\pi') = \widetilde{K}', \quad 1 \mapsto \pi'.$$

□

Now, on forgetting the rigidifications, we have the following exact sequence of Serre pro-algebraic groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{H}(\mathbf{S}) & \longrightarrow & \mathbf{Gr}(G) & \longrightarrow & \mathbf{Gr}(J) \longrightarrow 0, \\ & & \downarrow \gamma' & & \downarrow & & \parallel \\ 0 & \longrightarrow & < \mathcal{O}_X(D) > & \longrightarrow & \mathbf{Pic}^0(\mathbf{X}) & \xrightarrow{q} & \mathbf{Gr}(J) \longrightarrow 0 \end{array}$$

where  $\gamma'$  is given on  $k$ -rational sections by  $(\mathcal{O}_X(\frac{m}{d}D), a) \mapsto \mathcal{O}_X(\frac{m}{d}D)$ , and the vertical map in the middle is given by  $(\mathcal{L}, a) \mapsto \mathcal{L}$ . Hence, if we identify the kernel  $< \mathcal{O}_X(D) >$  of  $q$  with  $\mathbb{Z}/d\mathbb{Z}$  by sending  $\mathcal{O}_X(-D)$  to  $\bar{1} \in \mathbb{Z}/d\mathbb{Z}$ , we get the extension of pro-algebraic groups:

$$(80) \quad 0 \rightarrow \mathbb{Z}/d\mathbb{Z} \rightarrow \mathbf{Pic}^0(\mathbf{X}) \rightarrow \mathbf{Gr}(J) \rightarrow 0,$$

and this will give us an element of  $\text{Ext}^1(\mathbf{Gr}(J), \mathbb{Z}/d\mathbb{Z}) = \text{Ext}^1(\mathbf{Gr}(A'^0), \mathbb{Z}/d\mathbb{Z})$ .

By construction the map  $\gamma': \mathbf{H}(\mathbf{S}) \rightarrow < \mathcal{O}_X(D) > = \mathbb{Z}/d\mathbb{Z}$  is the composition of  $\mathbf{H}(\mathbf{S}) \rightarrow \pi_0(\mathbf{H}(\mathbf{S}))$  with  $-\gamma: \pi_0(\mathbf{H}(\mathbf{S})) \rightarrow \mathbb{Z}/d\mathbb{Z}$ . Hence the commutativity of (79) and (78) implies that the extension (80) coincides with the extension obtained by push-out along  $\mathbf{N}(\mathbf{K}) \rightarrow \pi_0(\mathbf{N}(\mathbf{K})) \cong \mathbb{Z}/d\mathbb{Z}$  followed by the pull-back along the canonical map  $\mathbf{Gr}(J) \rightarrow \mathbf{Gr}(A')$  of the lower exact sequence in (77). We can summarize these facts as follows:

**Proposition 5.3.2.** *With notations as above, the extension (80) in*

$$\text{Ext}^1(\mathbf{Gr}(J), \mathbb{Z}/d\mathbb{Z}) = \text{Ext}^1(\mathbf{Gr}(A'^0), \mathbb{Z}/d\mathbb{Z}) = \text{Hom}(\pi_1(\mathbf{Gr}(A')), \mathbb{Z}/d\mathbb{Z})$$

*coincides with the extension given by the morphism (64).*

**Corollary 5.3.3.** *Let  $n \in \mathbb{Z}_{\geq 1}$  be an integer. The morphism  $\Phi_n$  in (72) is an injective morphism of groups, which is an isomorphism if one of the following conditions is verified:*

- *The local field  $K$  is of mixed characteristic;*
- *The integer  $n$  is prime to  $p$ .*

*If one of the above conditions is satisfied, then  $\Phi_n$  coincides with Shafarevich's isomorphism (7) restricted to the  $n$  parts.*

*Proof.* In view of the previous Proposition and results in § 4 (Theorem 4.4.1 and Proposition 4.3.4) only the injectivity requires verification. Since, there is no non zero morphism from the connected pro-algebraic group  $\mathbf{Gr}(J)$  to a constant finite group, the canonical map (71) induces an injective maps between the group of extensions

$$\text{Ext}_k^1(\mathbf{Gr}(J), \mathbb{Z}/n'\mathbb{Z}) \rightarrow \text{Ext}_k^1(\mathbf{Gr}(J), \mathbb{Z}/n\mathbb{Z})$$

when  $n'|n$ . Hence, we only need to show that, for  $X_K$  a torsor under  $A_K$  of order  $d$ , the extension (70) is non zero in  $\mathrm{Ext}_k^1(\mathbf{Gr}(J), \mathbb{Z}/d\mathbb{Z})$  unless  $d = 1$ . Since the pro-algebraic group  $\mathbf{Pic}^0(\mathbf{X})$  is also connected, extension (70) is split if only if  $d = 1$ , and this fact implies that the torsor  $X_K$  is in fact trivial.  $\square$

**Remark 5.3.4.** The problem of extending the above Corollary to the  $p$ -parts in the equal characteristic case, reduces to showing that  $\Xi$  in Proposition 4.3.4 is always an isomorphism, for example, by checking that it explicates Shafarevich’s duality on the  $p$ -parts too. Although we have partial results in this direction, for example in the case of abelian varieties with totally degenerate reduction, a full answer is not yet at hand.

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